Lower Bounds for the Number of Small Convex k-Holes*

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Abstract

Let S be a set of n points in the plane in general position, that is, no three points of S are on a line. We consider an Erdős-type question on the least number $h_k(n)$ of convex k-holes in S, and give improved lower bounds on $h_k(n)$, for $3 \leq k \leq 5$. Specifically, we show that $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$, $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$, and $h_5(n) \geq \frac{3n}{4} - o(n)$.

1 Introduction

Let S be a set of n points in the plane in general position, that is, no three points of S lie on a common straight line. A *k-hole* of S is a simple polygon, P, spanned by k points from S, such that no other point of S is contained in the interior of P. A classical existence question raised by Erdős [8] is: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?". Esther Klein observed that every set of 5 points contains a convex 4-hole, and Harborth [12] showed that every set of 10 points determines a convex 5-hole. Both bounds are tight w.r.t. the cardinality of S. Only in 2007/08 Nicolás [14] and independently Gerken [11] proved that every sufficiently large point set contains a convex 6-hole. On the other hand, Horton [13] showed that there exist arbitrarily large sets which do not contain any convex 7-hole; see [1] for a brief survey.

A generalization of Erdős' question is: "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?". In this paper we concentrate on this question for $3 \leq k \leq 5$, that is, the number of empty triangles (3-holes), convex 4-holes, and convex 5-holes. We denote by $h_k(S)$ the number of convex k-holes determined by S, and by $h_k(n) = \min_{|S|=n} h_k(S)$ the number of convex k-holes any set of n points in general position must have. Throughout this paper let $\operatorname{ld} x = \frac{\log x}{\log 2}$ be the binary logarithm. Furthermore, we denote with $\operatorname{CH}(S)$ the convex hull of S and with $\partial \operatorname{CH}(S)$ the boundary of $\operatorname{CH}(S)$.

We start in Section 2 by providing improved bounds on the number of convex 5-holes. In particular, increasing the so far best bound $h_5(n) \geq \frac{n}{2} - O(1)$ [16] to $h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875$. In Section 3 we combine these results with a technique recently introduced by García [9, 10], and improve the currently best bounds on the number of empty triangles and convex 4-holes, $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ (both in [10]), to $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$, respectively.

2 Convex 5-holes

The currently best upper bound on the number of convex 5-holes, $h_5(n) \leq 1.0207n^2 + o(n^2)$ is by Bárány and Valtr [5], and it is widely conjectured that $h_5(n)$ grows quadratically. Still, to this date not even a super-linear lower bound is known.

As early as in 1987 Dehnhardt presented a lower bound of $h_5(n) \geq 3\lfloor \frac{n}{12} \rfloor$ in his thesis [6]. Unfortunately, this result, published in German only, remained unknown to the scientific community until recently. Thus, the best known lower bound was $h_5(n) \geq \lfloor \frac{n-4}{6} \rfloor$, obtained by Bárány and Károlyi [4]. In the presentation of [9] this bound was improved to $h_5(n) \geq \frac{2}{9}n - \frac{25}{9}$. A slightly better bound $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ was presented in [2], which was then sharpened to $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$ in [3]. The latest and so far best bound of $h_5(n) \geq \frac{n}{2} - O(1)$ is due to Valtr [16]. In this section we further improve this bound to $h_5(n) \geq \frac{3}{4}n - o(n)$.

We start by fine-tuning the proof from [3], showing $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$, by utilizing the results $h_5(10) = 1$ [12], $h_5(11) = 2$ [6], and $h_5(12) \geq 3$ [6]. Although this does not lead to an improved lower bound of $h_5(n)$ for large n, it provides better lower bounds for small values of n; see Table 1.

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n	10	11	12	13	14	15	16	17	18	19	20..23	24	25	26	27..30	31
$h_5(n)$	1	2	3	3..4	3..6	3..9	≥ 3	≥ 4	≥ 5	≥ 6	≥ 6	≥ 7	≥ 8	≥ 9	≥ 9	≥ 10
n	32	33	34..37	38	39	40	41..44	45	46	47	48..50	51	52	53	54	55..56
$h_5(n)$	≥ 11	≥ 12	≥ 12	≥ 13	≥ 14	≥ 15	≥ 15	≥ 16	≥ 17	≥ 18	≥ 18	≥ 19	≥ 19	≥ 20	≥ 21	≥ 21

Table 1: The updated bounds on $h_5(n)$ for small values of n.

Lemma 1 *Every set S of n points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1, 2, 3\}$) contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.*

Proof. Because of $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) \geq 3$ this is true for $m = 0$. Obviously $h_5(n) \geq h_5(n-1)$. Hence, $h_5(n) \geq 3$ for any $n \geq 12$.

If there exists a point $p \in ((\partial\,\mathrm{CH}\,(S)) \cap S)$ that is a point of a convex 5-hole, then $h_5(S) \geq 1 + h_5(S \backslash \{p\}) \geq 1 + h_5(n-1)$. In this case, the lemma is true by induction, as for $t = 1$ and $m > 0$, $h_5(n-1) = h_5(7 \cdot m + 9) \geq h_5(7 \cdot (m-1) + 9 + 3)$. (The case $t \in \{2, 3\}$ is trivial.)

Otherwise, each point $p \in ((\partial\,\mathrm{CH}\,(S)) \cap S)$ is not a point of a convex 5-hole. For $m > 0$ choose one such point p (e.g. the bottom-most one) and partition $S \backslash \{p\}$ (in clockwise order around p) into the following successive disjoint subsets: S_0 containing the first 7 points; S_0' containing the next 4 points; $(m-1)$ pairs of subsets: S_i containing 3 points and S_i' containing 4 points $(1 \leq i \leq (m-1))$; and the subset S_{rem} containing the remaining $(t+4)$ points. See Figure 1 for a sketch.

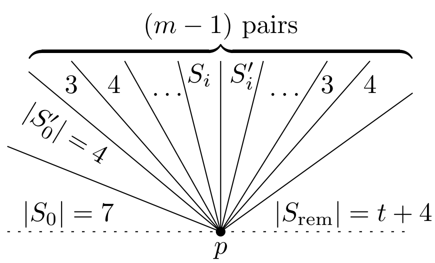

Figure 1: Partition of $S \backslash \{p\}$ clockwise around an extreme point p: starting with the pair S_0, S_0'; continuing with $(m-1)$ pairs of sets S_i, S_i', for $1 \leq i \leq (m-1)$, with $|S_i| = 3$ and $|S_i'| = 4$; and ending with the remainder set S_{rem}.

The subset $S_0 \cup S_0' \cup \{p\}$ has cardinality 12 and thus contains at least 3 convex 5-holes. The same is true for each subset $S_{i-1}' \cup S_i \cup S_i' \cup \{p\}$ $(1 \leq i \leq (m-1))$. Finally, the subset $S_{m-1}' \cup S_{\text{rem}} \cup \{p\}$ has cardinality $(9+t)$ and therefore contains at least t convex 5-holes. Note that we count every convex 5-hole at most once, as the considered subsets of 10, 11, and 12 points, respectively, overlap in at most 4 points. In total this gives at least $3 + (m-1) \cdot 3 + t = 3 \cdot \frac{n-9-t}{7} + t = \frac{3n-27+4t}{7}$ convex 5-holes. □

Corollary 2 *Every set S of 17 points in the plane in general position contains at least $h_5(17) \geq 4$ convex 5-holes.*

Table 1 shows the bounds on $h_5(n)$ obtained by Lemma 1, for some small values of n. By Harborth [12] $h_5(10) = 1$, and by Dehnhardt [6] $h_5(11) = 2$ and $h_5(12) \geq 3$. The bounds for $n = 51$ and for $57 \leq n < 62250$ (not shown in the table) are due to $h_5(n) \geq \lceil \frac{n}{2} \rceil - 7$ from Valtr [16]. The bounds $h_5(12) \leq 3$, $h_5(13) \leq 4$, $h_5(14) \leq 6$, and $h_5(15) \leq 9$ are from [3, 17].

In the following theorem we present an improved lower bound on $h_5(n)$ for larger n.

Theorem 3 *Every set S of $n \geq 12$ points in the plane in general position contains at least $h_5(n) \geq \frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} = \frac{3n}{4} - o(n)$ convex 5-holes.*

Proof. For $12 \leq n < 17$ we count three convex 5-holes for S. For $17 \leq n < 24$ we can count four convex 5-holes for S by Corollary 2.

If $n \geq 24$ consider an (almost) halving line ℓ of S which splits S into S_L ($|S_L| = \lceil \frac{n}{2} \rceil$) and S_R ($|S_R| = \lfloor \frac{n}{2} \rfloor$) and does not contain any point of S. See Figure 2.

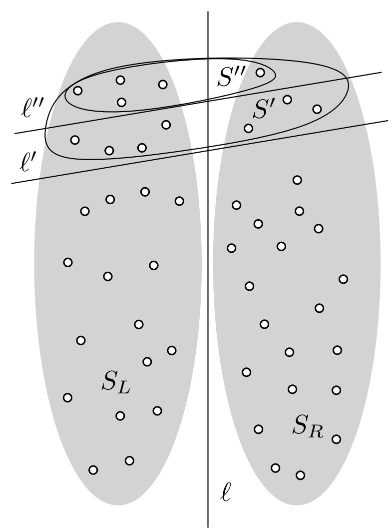

Figure 2: A point set S split by a halving line ℓ into two point sets $S_L, S_R \subset S$. The line ℓ' cuts off a set $S' \subseteq S$, consisting of 8 points of S_L and 4 points of S_R. The line ℓ'' is parallel to ℓ' and halves $S_L \cap S'$.

Furthermore, consider a line ℓ' that intersects ℓ and cuts off a set $S' \subseteq S$, consisting of eight points from S_L

and four points from S_R. That this is in fact possible is folklore, see e.g. Exercise 4.5 (b) in [7]. Let a line ℓ'' be parallel to ℓ' and split $S' \cap S_L$ into two groups of four points, and let $S'' \subset S'$ be the set which is cut off by ℓ''. Note that neither ℓ' nor ℓ'' contain any points of S.

As $|S'| = 12$ we have that S' contains at least three convex 5-holes. We distinguish two cases.

Case 1: S' contains at least three convex 5-holes which are not intersected by ℓ. Then each of these 5-holes contains only points from S_L and thus at least one point above ℓ''. We count the three convex 5-holes for the set S_L and continue on $S \backslash S''$.

Case 2: S' contains at most two convex 5-holes which are not intersected by ℓ. Then at least one convex 5-hole in S' is intersected by ℓ. We count one convex 5-hole for the halving line ℓ and continue on $S \backslash S'$.

Note that in both cases we cut off at least four points from S_L, but at most four points from S_R. Thus, we can repeat this process until we have processed all $\lceil \frac{n}{2} \rceil$ points of S_L. Let c_L be the number of convex 5-holes counted for ℓ when processing S_L. Hence, Case 2 appeared c_L times, and Case 1 appeared at least $\lfloor \frac{1}{4} \cdot (\lceil \frac{n}{2} \rceil - 8c_L) \rfloor - 1$ times. Therefore, the number of convex 5-holes we counted in S_L (i.e., not intersecting ℓ) is $h_5(S_L) \geq 3 \left(\lfloor \frac{1}{4} \left(\lceil \frac{n}{2} \rceil - 8c_L \right) \rfloor - 1 \right)$.

Repeating the same procedure for S_R (exchanging the roles of S_L and S_R), we obtain $h_5(S_R) \geq 3 \left(\lfloor \frac{1}{4} \left(\lfloor \frac{n}{2} \rfloor - 8c_R \right) \rfloor - 1 \right)$, where c_R is the number of convex 5-holes which we counted for ℓ when processing S_R. Note that any convex 5-hole intersected by ℓ, which we counted while processing S_L, might have occurred again when processing S_R. Thus, the total number c of convex 5-holes intersected by ℓ is at least $\max\{c_L, c_R\} \geq \frac{c_L + c_R}{2}$. As $h_5(S) = h_5(S_L) + h_5(S_R) + c$, we obtain

$$\begin{aligned} h_5(S) \geq\; & 3 \cdot \left(\left\lfloor \frac{1}{4} \cdot \left(\left\lceil \frac{n}{2} \right\rceil - 8c_L \right) \right\rfloor - 1 \right) \\ & + 3 \cdot \left(\left\lfloor \frac{1}{4} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right\rfloor - 1 \right) + \frac{c_L + c_R}{2} \ . \end{aligned}$$

Considering that

$$\left\lfloor \frac{\lceil \frac{n}{2} \rceil}{4} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{4} \right\rfloor = \begin{cases} 2 \cdot \left\lfloor \frac{\frac{n}{2}}{4} \right\rfloor & \ldots\ n \text{ is even} \\ \left\lfloor \frac{\frac{n+1}{2}}{4} \right\rfloor + \left\lfloor \frac{\frac{n-1}{2}}{4} \right\rfloor & \ldots\ n \text{ is odd} \end{cases}$$

is $\geq \frac{n}{4} - \frac{6}{4}$ in both cases, careful transformation gives

$$h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \tag{1}$$

as a first lower bound for the number of convex 5-holes in S. Using $h_5(S) = c + h_5(S_L) + h_5(S_R)$, and the fact that the (almost) halving line ℓ splits S such that $|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$, we get $h_5(S) \geq \frac{c_L + c_R}{2} + h_5(\lceil \frac{n}{2} \rceil) + h_5(\lfloor \frac{n}{2} \rfloor) \geq \frac{c_L + c_R}{2} + h_5(\lceil \frac{n-1}{2} \rceil) + h_5(\lceil \frac{n-1}{2} \rceil)$, and hence, a second lower bound for $h_5(S)$:

$$h_5(S) \geq \frac{c_L + c_R}{2} + 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \ . \tag{2}$$

Combining this with the bound (1), we obtain

$$\begin{aligned} h_5(S) \geq \max \Bigg\{ & \left(\frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \right), \\ & \left(\frac{c_L + c_R}{2} + 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \right) \Bigg\} \ . \end{aligned} \tag{3}$$

Note that the first term in inequality (3) is strictly monotonically decreasing in $\frac{c_L + c_R}{2}$, while the second term is strictly monotonically increasing in $\frac{c_L + c_R}{2}$. Thus, the minimum of the lower bound in (3) is reached if both bounds are equal.

$$\begin{aligned} \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} &= \frac{c_L + c_R}{2} + 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \\ \frac{3n}{4} - \frac{21}{2} - 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) &= 12 \cdot \frac{c_L + c_R}{2} \\ \frac{c_L + c_R}{2} &= \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \end{aligned}$$

Plugging this result for $\frac{c_L + c_R}{2}$ into the lower bound (2) for $h_5(S)$, we obtain a lower bound for $h_5(S)$ for any S with n points. Therefore, this also leads to a lower bound for $h_5(n)$.

$$\begin{aligned} h_5(n) &\geq \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) + 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \\ &= \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \ . \end{aligned} \tag{4}$$

We show by induction that this recursion resolves to $h_5(n) \geq \frac{3n}{4} - n^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8}$, for $n \geq 12$. We know that $h_5(12), \ldots, h_5(16) \geq 3$ and $h_5(17), \ldots, h_5(23) \geq 4$ (see first paragraph of this proof). As $\frac{3n}{4} - n^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8}$ is monotonically increasing for $12 \leq n \leq 23$, it is sufficient to check the induction base for $n = 16$ and $n = 23$: $h_5(16) \geq 3 \geq 2.578 \geq \frac{3 \cdot 16}{4} - 16^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8}$ and $h_5(23) \geq 4 \geq 3.609 \geq \frac{3 \cdot 23}{4} - 23^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8}$. For $n \geq 24$ we insert the claim into the recursive formula:

$$\begin{aligned} h_5(n) &\geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) \\ &\geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot \left(\frac{3 \frac{n-1}{2}}{4} - \left(\frac{n-1}{2} \right)^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8} \right) \\ &= \frac{3n}{4} + \frac{15}{8} - \frac{11}{6} \cdot \frac{1}{2^{\mathrm{ld}\, \frac{11}{6}}} \cdot (n-1)^{\mathrm{ld}\, \frac{11}{6}} \\ &\geq \frac{3n}{4} - n^{\mathrm{ld}\, \frac{11}{6}} + \frac{15}{8} \ . \end{aligned}$$

The last inequality is true because $(n-1)^{\mathrm{ld}\, \frac{11}{6}} < n^{\mathrm{ld}\, \frac{11}{6}}$.

This proves the claim and the theorem as we have: $h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875 = \frac{3n}{4} - o(n)$. $\square$

3 Empty triangles and convex 4-holes

For this section we are going to use some definitions and notation used in [15, 9, 10]. Let S be a set of n points in the plane in general position. We need to define a total order on the points of S. In addition, this order has to define a line ℓ_q through every point $q \in S$, such that each point $r \in S$ is either in the closed halfplane "below" ℓ_q, i.e., $q \geq r$, or in the open halfplane "above" ℓ_q, i.e., $q < r$. In [10] the points of S are sorted in increasing order of the ordinate y (with the additional restriction that no two points have equal ordinate). Observe though, that of course any direction is a valid order for the points of S. Furthermore, observe that also a cyclic order around some point $p \in ((\partial \operatorname{CH}(S)) \cap S)$ is a valid order for the points of $S \backslash \{p\}$, as there exists a line ℓ through p, such that all points of $S \backslash \{p\}$ are in an open halfplane bounded by ℓ. This will be crucial for the proof of Lemma 6 where we will order the points of a set $S \backslash \{p\}$ around such a point p. Note that, because of the general position assumption for S, no two points in $S \backslash \{p\}$ are equivalent in this order. Anyhow, for simplicity, and apart from the aforementioned exception, we will use the order along the ordinate of S, as in [10].

Let P be a convex 5-hole spanned by points of S and let v be the *top vertex* of P, i.e., the vertex of P with highest order. We name an empty triangle *generated by* P if it is spanned by v and the two vertices of P that are not adjacent (on the boundary of P) to v. Let $h_{3|5}(S)$ be the number of such triangles determined by S, and let $h_{3|5}(n) = \min_{|S|=n} h_{3|5}(S)$ be the number of empty triangles generated by convex 5-holes that every set of n points spans at least. Likewise, we name a convex 4-hole *generated by* P if it is spanned by all vertices of P except for one of the two vertices of P that are adjacent (on the boundary of P) to v. Observe that each convex 5-hole generates two convex 4-holes by this definition. Let $h_{4|5}(S)$ be the number of such 4-holes determined by S, and let $h_{4|5}(n) = \min_{|S|=n} h_{4|5}(S)$ be the number of convex 4-holes generated by convex 5-holes that every set of n points spans at least.

García [10] recently proved that $h_3(S) = n^2 - 5n + H + 4 + h_{3|5}(S) \geq n^2 - 5n + H + 4 + h_{3|5}(n)$ and $h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S) \geq \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(n)$, where H is the number of points of $((\partial \operatorname{CH}(S)) \cap S)$. Consequently, this gives $h_3(n) \geq n^2 - 5n + 7 + h_{3|5}(n)$ and $h_4(n) \geq \frac{n^2}{2} - \frac{7n}{2} + 6 + h_{4|5}(n)$, as $H \geq 3$. Observe that this implies that $h_{3|5}(S)$ and $h_{4|5}(S)$ (and of course $h_{3|5}(n)$ and $h_{4|5}(n)$) do not depend on the chosen order of the points. As changing the order does not change the point set, $h_3(S)$ and $h_4(S)$ are of course independent of the order. Furthermore, García proved that the number of empty triangles (or convex 4-holes) not generated by convex 5-holes is an invariant of the point set. Hence, although the empty triangles and convex 4-holes generated by convex 5-holes may change with different orders, their numbers stay the same.

Proving $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-4}{8} \right\rfloor$ and $h_{4|5}(n) \geq 6 \cdot \left\lfloor \frac{n-4}{8} \right\rfloor$, García presented the improved bounds $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$. We will improve these bounds on $h_{3|5}(n)$ and $h_{4|5}(n)$. Showing that for each convex 5-hole counted in Lemma 1 we may count one empty triangle generated by convex 5-holes and two convex 4-holes generated by convex 5-holes will already give an improved bound for both, $h_{3|5}(n)$ and $h_{4|5}(n)$. But using a slightly adapted version of the proof from Theorem 3 will improve the bound on $h_{4|5}(n)$ even further. To this end we have to first prove the base case, i.e., sets of 10, 11, and 12 points.

Having a close look at the example shown in Figure 3, one can see that as soon as the triangle $\triangle$ (or the convex 4-hole $\diamond$) is generated by more than one convex 5-hole, there must exist at least one convex 6-hole. We state this fact in more detail and prove it in the following lemma. Note that a similar approach and figure has been used in [10].

Lemma 4 *Let S be a set of $n \geq 6$ points in the plane in general position. Let $\triangle$ ($\diamond$) be an empty triangle (a convex 4-hole) of S. If $\triangle$ ($\diamond$) is generated by at least two convex 5-holes, $⬠_1$ and $⬠_2$, of S, then there exists at least one convex 6-hole, $⬡_1$, of S, containing $⬠_1$, and one convex 6-hole, $⬡_2$, of S, containing $⬠_2$, where $⬡_1 = ⬡_2$ is possible.*

Proof. See Figure 3 (top). Assume that there exists at least one empty triangle, $\triangle = \langle p_i, p_j, p_k \rangle$, with p_k being the top vertex, that is generated by two different convex 5-holes. Let one of them, $⬠_1$, be spanned by the points p_i, p_j, p_L, p_k, p_R (the points shown as full dots in the figure). As $\triangle$ is generated by another convex 5-hole, $⬠_2$, there must be at least one additional point in one of the regions L_h, L_l, R_h, and R_l. Otherwise, the new pentagon would not be empty, not be convex, or $\triangle$ would not be generated by it (recall that p_k must be the highest point). W.l.o.g. assume that there exists at least one point p_{new} in R_l. It is easy to see that in this case there exists a convex 4-hole spanned by the points p_i, p_k, p_R, p'_R ($p'_R = p_{new}$ is possible, but not necessary). Together with p_j and p_L this forms a convex 6-hole which contains $⬠_1$. Starting the argument with $\triangle$ being generated by $⬠_2$, proves that also $⬠_2$ is contained in a convex 6-hole.

The argumentation is analogous for a convex 4-hole, $\diamond$, that is generated by two different convex 5-holes. See Figure 3 (bottom). The only difference to the previous case (with $\triangle$) is that the additional point p_{new} can not exist in either L_l or R_l, depending on which convex 4-hole (either $\diamond = \langle p_i, p_j, p_L, p_k \rangle$ or $\diamond = \langle p_i, p_j, p_k, p_R \rangle$) is considered. The former situation is depicted in Figure 3 (bottom). $\square$

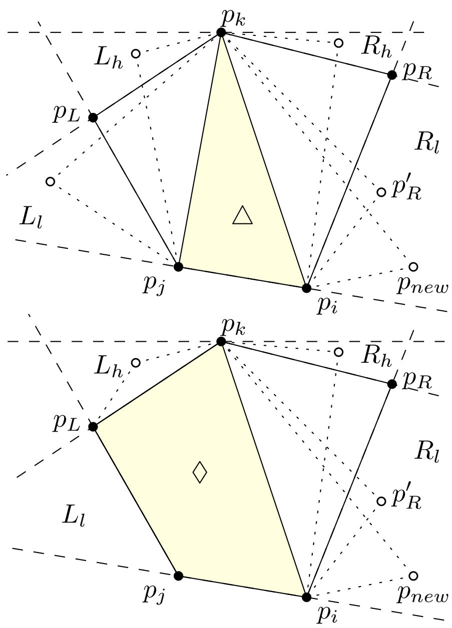

Figure 3: Auxiliary figure for the proof of Lemma 4.

Using Lemma 4 we are able to provide the base cases $10 \leq n \leq 12$ for $h_{3|5}(n)$ and $h_{4|5}(n)$. The proof is omitted in this extended abstract.

Lemma 5 *Every set of 10, 11, or 12 points in the plane in general position contains (i) at least 1, 2, and 3, respectively, different empty triangles generated by convex 5-holes (i.e., $h_{3|5}(10) = 1$, $h_{3|5}(11) = 2$, and $h_{3|5}(12) = 3$) and (ii) at least 2, 4, and 6, respectively, different convex 4-holes generated by convex 5-holes (i.e., $h_{4|5}(10) = 2$, $h_{4|5}(11) = 4$, and $h_{4|5}(12) = 6$).*

These base cases allow a lemma similar to Lemma 1. The proof follows the lines of the proof of Lemma 1 and is omitted in this extended abstract.

Lemma 6 *Every set S of n points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1, 2, 3\}$) contains at least $h_{3|5}(n) \geq \frac{3n-27+4t}{7}$ empty triangles generated by convex 5-holes and at least $h_{4|5}(n) \geq 2 \cdot \frac{(3n-27+4t)}{7}$ convex 4-holes generated by convex 5-holes.*

As mentioned above, this lemma already improves the bounds for $h_{3|5}(n)$ and $h_{4|5}(n)$. We will further improve the bound for $h_{4|5}(n)$ in Theorem 8. In the following theorem we state only the bound for $h_{3|5}(n)$.

Theorem 7 *Every set S of $n \geq 12$ points in the plane in general position contains at least $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-12}{7} \right\rfloor + 3 + f(|S_{rem}|) \geq \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles generated by convex 5-holes. The point set $S_{rem} \subset S$ is the remainder set with $0 \leq |S_{rem}| \equiv (n-12) \mod 7 \leq 6$, and $f(0 \ldots 4) = 0$, $f(5) = 1$, and $f(6) = 2$.*

Proof. The first inequality in the bound, $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-12}{7} \right\rfloor + 3 + f(|S_{\text{rem}}|)$, is simply a reformulation of the bound in Lemma 6. The second inequality results from taking the minimum of the first inequality over all possible values for $|S_{\text{rem}}|$. (This minimum is obtained by $|S_{\text{rem}}| = 4$.) $\square$

The basic principles of the proof of the following theorem are the same as in the proof of Theorem 3. The main difference is that, for excluding over-counting, a slightly different counting is needed. The proof is omitted in this extended abstract and we only state the result.

Theorem 8 *Every set S of $n \geq 12$ points in the plane in general position contains at least $h_{4|5}(n) \geq \frac{5n}{4} - \frac{383}{303} \cdot n^{\mathrm{ld}\,\frac{19}{10}} + \frac{55}{24} = \frac{5n}{4} - o(n)$ convex 4-holes generated by convex 5-holes.*

Remark: To use the principles of the proof of Theorem 3 also for empty triangles generated by convex 5-holes, a very disadvantageous splitting is necessary to avoid over-counting. This would lead to a bound inferior to the one from Theorem 7.

Recall that García [10] recently proved $h_3(S) \geq n^2 - 5n + H + 4 + h_{3|5}(n)$ and $h_4(S) \geq \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(n)$. Combining these results with Theorem 7 and Theorem 8 we can state the following corollary.

Corollary 9 *Every set S of $n \geq 12$ points in the plane in general position and with H points on the boundary of its convex hull contains at least $h_3(S) \geq n^2 - 5n + H + 4 + \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles and at least $h_4(S) \geq \frac{n^2}{2} - \frac{9n}{4} - \frac{383}{303} \cdot n^{\mathrm{ld}\,\frac{19}{10}} + H + \frac{127}{24}$ convex 4-holes. Consequently, $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$.*

4 Conclusion

In this paper we improved the lower bounds on the least number $h_k(n)$ of convex k-holes any set of n points contains, for $3 \leq k \leq 5$. The question whether there exists a super-linear lower bound for the number of convex 5-holes remains unsettled, though.

Still, we are able to answer two questions that Dehnhardt [6] asked in 1987. Already in [3] a set of 12 points containing only three convex 5-holes has been presented, implying $h_5(12) = 3$. This disproved Dehnhardt's conjecture of $h_5(12) = 4$. Recall that we know from García [10], that $h_3(S) = n^2 - 5n + H + 4 + h_{3|5}(S)$ and $h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S)$, where $h_{3|5}(S)$ ($h_{4|5}(S)$) is the number of empty triangles (convex 4-holes) generated by convex 5-holes in S.

Consider the set S_{12} with $n = 12$ points and $H = 3$, depicted in Figure 4. It can be easily checked that

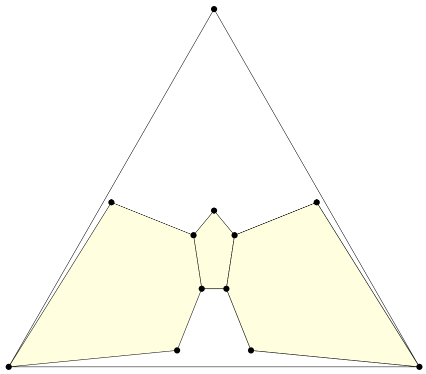

Figure 4: Set of 12 points with triangular convex hull, generating the minimal number of 3-holes (94), convex 4-holes (42), and convex 5-holes (3). The coordinates (x, y) of the 12 points are: (0, 0); (100, 0); (50, 87); (50, 38); (55, 32); (53, 19); (47, 19); (45, 32); (41, 4); (59, 4); (25, 40); (75, 40).

this point set contains only the 3 shown convex 5-holes. Hence, $h_{3|5}(S_{12}) = 3$ and $h_{4|5}(S_{12}) = 6$, as by Lemma 5 $h_{3|5}(12) = 3$ and $h_{4|5}(12) = 6$. Inserting into the above equations, we get $h_3(S_{12}) = h_3(12) = 144-60+3+4+3 = 94$ and $h_4(S_{12}) = h_4(12) = 72-42+3+3+6 = 42$, as $h_3(12) \geq 94$ and $h_4(12) \geq 42$ (by [6]). Of course, $h_3(S_{12})$ and $h_4(S_{12})$ can also be derived by counting all empty triangles and convex 4-holes in S_{12}. This disproves two conjectures of Dehnhardt in [6], namely $h_3(12) = 95$ and $h_4(12) = 44$.

Furthermore, his question for a set of n points that minimizes at least one of $h_3(n)$, $h_4(n)$, and $h_5(n)$, but not all of them is answered by the set of 12 points presented in [3], which has only 3 convex 5-holes but contains (non-minimal) 95 empty triangles and (non-minimal) 43 convex 4-holes.

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