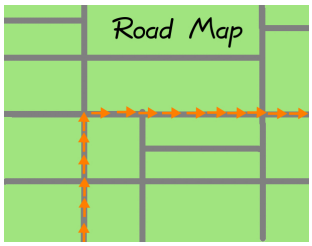
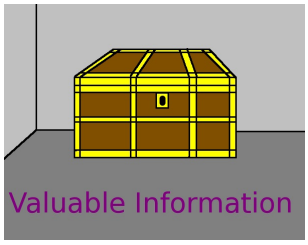
Module 0.3: Order of Operations and Calculator Skills



This module is intended only for those students who had difficulty with Diagnostic One. The module has two purposes:

- For any student using this book, it is surely not their first math course, and so they have been taught the "order of operations" before. However, especially when there have been several months or even years since the previous math class, some students may have forgotten the order of operations—this module can help such a student get "up to speed." The problems in this module represent a cornucopia of different formulas taken from inside this book—therefore, you are practicing with very realistic formulas taken from finance, science, and economics. (This is in contrast with learning from some collection of difficult arbitrary mathematical challenges that were constructed only to confuse you, as would have been popular in mid-to-late twentieth-century textbooks.)
- The second purpose is for students who might be switching from one category of calculator to another. For example, I personally do not permit graphing calculators, but instead use scientific calculators in my classroom; therefore, a student unaccustomed to scientific calculators can use a few practice problems to ensure that they understand exactly how to key-in any specific calculation. Also, one of the main differences between graphing calculators and scientific calculators has to do with how they handle the order of operations—all the more reason that this exercise shall be fruitful for students switching from one type of calculator to another.



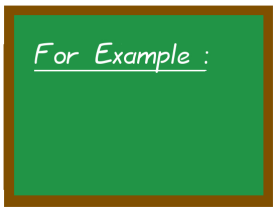
There are many types of calculators out there and so it isn't possible for a textbook (or an instructor) to be able to know them all. However, your calculator's manual is an excellent resource for finding out how to key things in, or to figure out what button you need in order to do a particular task.

Don't despair if you have thrown away the manual that came with your calculator long ago. Almost every student throws that away. The websites of the calculator manufacturers all have downloadable electronic copies of those manuals. These have the added advantages of not taking up space in your school bag, and also being rapidly searchable via the "find" command.



Mathematical operations have something called "order of precedence" or the "order of operations." In any calculation, you must remember to do the steps in the correct order. However, luckily for all students, there are many little limericks to help anyone remember the order. I've listed several below. Once you do a few drills, this ordering will be burned into your brain, and you will forever be unlikely to make a mistake of this kind. But first, I'd like you to consider an example from ordinary daily life.

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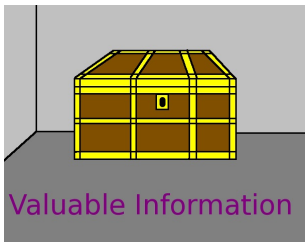


0-3-1

Many times, the order that you do things in doesn't matter. However, sometimes it matters a great deal! Let's consider an example:

A friend of mine, before going to work, would always have a glass of orange juice. This was often the last thing she did before driving away from home to the office. One morning in the summer of 2001, she was tired. The normal sequence of operations is 1) Open refrigerator, 2) Remove orange-juice carton, 3) Close refrigerator, 4) Shake cartoon vigorously, 5) Open the carton, 6) Pour juice, 7) Close carton, *et cetera*...

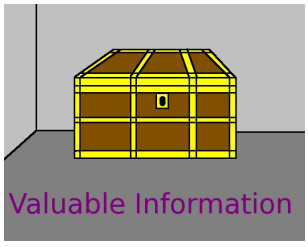
This particular morning, however, she interchanged steps 5 and 4. It is a very small error. Since the carton was open when she shook it vigorously, she was covered head to toe in orange juice. This means that she had to shower again, change clothes, and so on, and lost about 45 minutes of her day, for a tiny mistake. She wasn't stupid by any means—in fact, she has a PH.D. in computer science—rather she was just careless about the sequence of the operations comprising the act of pouring orange juice.



The "order of precedence" or "order of operations" is given as follows:

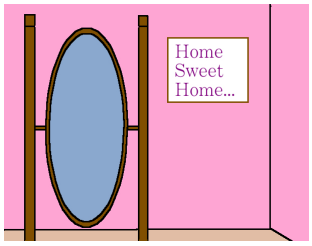
Operation	Rank	Nickname
Parentheses	1^{st}	Please
Roots Exponents	2^{nd}	Remember Everyday
Multiplication Division	3^{rd}	My Dear
Addition Subtraction	4^{th}	Aunt Sally
Fraction Bar	5^{th}	Fredrickson

This means that you do the 1^{st} class operations first, then the 2^{nd} class ones, then 3^{rd} class ones, followed by 4^{th} class ones, and lastly, the 5^{th} class ones. A bunch of examples will make this *much* more clear—but first, I'll share with you a few alternative limericks.



- Please Remember (that) Extremely Dilatory Math Students Always Fail.
- Plenty (of) Religious Extremists Don't Master Any Scientific Facts.
- Practicing Real Exercises (in) Mathematics Does Always Seem Fun.
- Plenty (of) Really Egotistical Millionaires Don't Savor Artisanal Food.
- Powerful Reasons (for) Exploring Math Do (not) Supply Adequate Fear.
- Pretty Expensive Reasons Mortgage Dealers Suck Are Foreclosures.
- President Reagan Enlarged (the) Monetary Deficit Substantially, About (three)-Fold.

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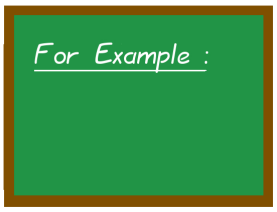


A Pause for Reflection...

As I said on Page 59, the heart of this module will be a sequence of real-world formulas, each of which is coming up again later in this book. Most of them are related to finance or economics, but I also have three formulas from high-school geometry to ease our journey toward harder formulas.

We will practice both our calculator skills and the order of operations by plugging data into those formulas, and seeing if we can get the right answer to come out. We will see that we truly must obey the order of operations. If we do not obey the order of operations, then we get wrong answers.

Before we do that, however, I regret that we must very briefly practice with one drill in the twentieth-century style. Think of it as a short warmup of jumping jacks or push-ups prior to a fun athletic game.



0-3-2

Let us suppose you are asked to compute

$$(-3) \times (-6) + 3 \text{ as well as } 4 \times (-3) + (-7)$$

on an exam. How do we proceed?

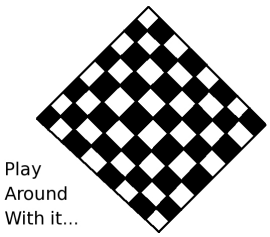
First, we see that both computations have one multiplication and one addition. Since multiplication has precedence over addition (3rd rank versus 4th rank), we are obligated to do the multiplication first. There are no exponents here (2nd rank) and the parentheses (1st rank) have just a single number in them, so there's nothing to be done with that.

Okay, so we do the multiplication first getting

$$(-3) \times (-6) + 3 = 18 + 3 = 21$$

and then for the other one

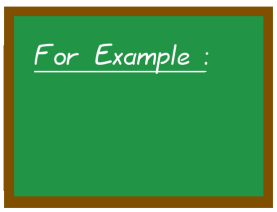
$$4 \times (-3) + (-7) = -12 + -7 = -19$$



0-3-3

Well that wasn't so bad! Try it yourself, now, before continuing further.

- Compute $6 + (-8) \times (-2)$. [Answer: 22.]
- Compute $(-17) - (-6) + (-3)$. [Answer: -14.]
- Compute $(-5) \times (-8) \times (-9)$. [Answer: -360.]
- Compute $-5 + (-8) \times (14)$. [Answer: -117.]



0-3-4

Let's imagine that some homework problem asks you to compute the area of a circle. A quick internet search reveals that the area of a circle is given by $A = \pi r^2$, where r is the radius. Suppose $r = 3$ and recall that

$$\pi = 3.14159265358979323846264\cdots$$

Now we have $A = 3.14159\cdots \times 3^2$. The exponent (2nd rank) has highest priority, because there are no parentheses (1st rank) here. We know $3^2 = 9$ and so we have $A = 3.14159\cdots \times 9$, and our calculator tells us that $A = 28.2743\cdots$.

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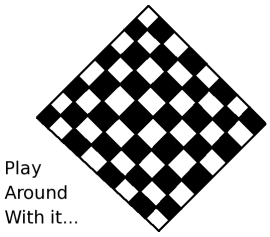


As you can see, our first example consisted of a squaring and a multiply. The squaring is a type of exponent, and it had to come first. We have no right to do the multiplication first because exponents have precedence over multiplications. In fact, if we did the multiplication first, we'd be squaring $9.42477\cdots$ and then we'd get $A = 88.8264\cdots$, which is very far off—not very close at all.

By the way, there's no reason to memorize the digits of π. Your calculator is very likely to have a π button. You should check your calculator's manual for details.



When I was in late elementary and early middle school (1986–1990), it was fashionable for math teachers to hold competitions to see which students could memorize longer and longer sequences of the digits of π. It is difficult to imagine what kind of impractical mind—gravely out of touch with reality—would imagine that this is a productive use of a young child's time. I am very happy that π-memorizing contents have fallen out of favor.

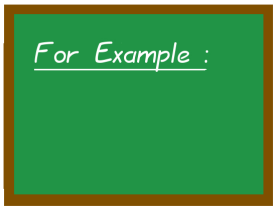


0-3-5

With the following data, use the formula $A = \pi r^2$ to find the area of a circle.

- If $r = 5$ then what is A? [Answer: $78.5398\cdots$.]
- If $r = 28$ then what is A? [Answer: $2463.00\cdots$.]
- If $r = 0.1$ then what is A? [Answer: $0.0314159\cdots$.]

There is a cute historical anecdote about π at the end of this module, on Page 84. Also, an entire book has been written about π, entitled *Pi Unleashed* by Jörg Arndt, Christoph Haenel, C. Lischka and D. Lischka, published by Springer-Verlag in 2001. It is a very interesting and readable book.



0-3-6

Now suppose you have a homework problem that discusses a cylinder. We'll actually look at an economics problem involving a cylinder on Page 1169, where a company has to determine what sizes of a cylinder would require the least amount of metal (and therefore be cheapest), when mass producing oil barrels. A quick internet search reveals that the volume of a cylinder is given by

$$V = \pi r^2 h$$

where you now know what π is, and r is the radius, while h is the height. Let's try using this. If the height of a can is 20 cm and the radius is 5 cm, then we have

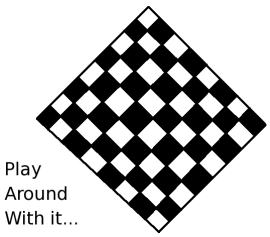
$$\begin{aligned} V &= \pi(5)^2 20 \\ &= \pi(25)(20) \\ &= \pi 500 \\ &= 1570.79\cdots \end{aligned}$$

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You are to notice several things from the previous example. First, that we did the exponent (raising 5 to the second power) before either of the multiplications (by 20 and by π). That's because the exponentiation takes precedence over the multiplication. However, while I did the multiplication by 20 before the multiplication by π, it would have been permissible to do the multiplications in the reverse order. That's because they are of equal precedence (3rd rank).

If I did reverse the order of the multiplications, I'd have multiplied 25 and π to get $78.5398\cdots$ and then multiplied that in turn by 20 to get $1570.79\cdots$ as desired.



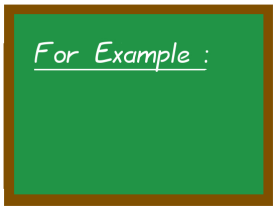
0-3-7

Try the formula for the volume of a cylinder ($V = \pi r^2 h$) with the following data.

- What if $r = 4$ and $h = 8$? [Answer: $402.123\cdots$.]
- What if $r = 8.44587$ and $h = 16.8917$? [Answer: $3785.40\cdots$.]
- If you have gotten the first two correct, you can skip the third one.
- What if $r = 10$ and $h = 1.5$? [Answer: $471.238\cdots$.]

In a proper geometry course, we would have reason to ask you to write answers in the form 128π and 150π, which happen to be the first and last bullet above. However, our visits with geometry in this textbook will be extremely brief and since decimal notation is preferred throughout economics and finance, we will use decimal notation instead.

By the way, now would be a good time to emphasize that you do not need to memorize (or even understand) any of these formulas at all, at this time. Just put the given numbers in, and try to get the correct answer to come out.



0-3-8

One last geometry problem and we can leave geometry behind and switch to financial and economic problems. Let's suppose you have a homework problem about a sphere. A quick internet search reveals that the volume of a sphere is

$$V = \left(\frac{4}{3}\right)\pi r^3$$

and suppose we are told that $r = 5$. Then we can find V.

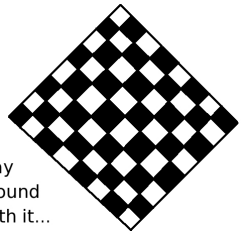
The operation with the most precedence is the parentheses (1st rank). We first calculate $(4/3) = 1.3\overline{3}$. Next, we do the exponent, since that has highest precedence (2nd rank) of what is left. The calculator says that $5^3 = 125$.

Lastly, we can do the multiplications. We can do these in any order by the way, a property called "the commutative law of multiplication" in more formal mathematical textbooks. In any case, our calculator tells us

$$1.33333\cdots \times 3.14159\cdots \times 125 = 523.598\cdots$$

which is the volume of the sphere.

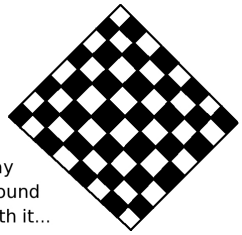
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0-3-9

Try the $V = \frac{4}{3}\pi r^3$ formula yourself with the following data.

- What if $r = 6$? [Answer: $V = 904.778\cdots$.]
- What if $r = 7$? [Answer: $V = 1436.75\cdots$.]
- What if $r = 8$? [Answer: $V = 2144.66\cdots$.]



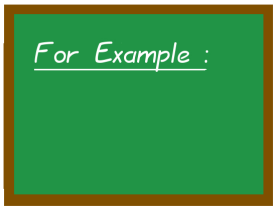
0-3-10

Another way to write that formula is

$$V = \frac{4\pi r^3}{3}$$

which, it turns out, is equivalent.

- What do you get for $r = 5$ with the new formula? [Answer: $523.598\cdots$.]
- What do you get for $r = 6$ with the new formula? [Answer: $904.778\cdots$.]



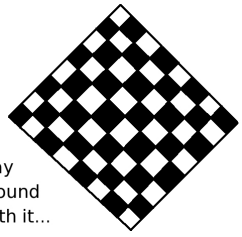
0-3-11

On Page 254 you will learn the formula for simple interest, which is $A = P(1+rt)$. Let's use the order of operations to evaluate this formula when $P = 2500$, and $r = 0.05$ but $t = 0.5$. We would start with

$$A = 2500(1 + 0.05 \times 0.5)$$

and since parenthesis have the most precedence (1st rank), we must evaluate $(1+0.05\times 0.5)$ before we do anything else.

Inside the parentheses, there is an addition and a multiplication too. The multiplication has higher precedence (3rd rank versus 4th rank), so we do $0.05 \times 0.5 = 0.025$. Next comes the addition, which is $1+0.025 = 1.025$. Now we have $A = 2500 \times 1.025$, and the only thing left is the multiplication: $2500 \times 1.025 = 2562.5$.



0-3-12

Now plug in the following data yourself.

- $P = 500$, $r = 0.08$ and $t = 1/4$. [Answer: \$ 510.00.]
- $P = 100,000$, $r = 0.03$ and $t = 1/2$. [Answer: \$ 101,500.00.]
- $P = 2000$, $r = 0.12$ and $t = 3$. [Answer: \$ 2,720.00.]

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For Example :

0-3-13

Let's imagine that, for some unfortunate reason, you are asked to compute

$$S = 3 + 13 + 23 + 33 + 43 + 53 + \cdots + 123$$

There is a shortcut formula,

$$S = \frac{n(a+z)}{2}$$

called "the sum of an arithmetic progression." The value of a is the first number present, in this case 3. Meanwhile, $z = 123$ is the last number present and $n = 13$, the number of members of the sum. (Trust me, there are some genuinely practical applications of this formula, but let's not worry about that just now.) We now have

$$S = \frac{13(3+123)}{2} = \frac{(13)(126)}{2} = \frac{1638}{2} = 819$$

As you can see, we were obligated to do the parentheses first, as they have highest precedence (1st rank). Once we had added 3 and 123 to get 126, then we have to multiply by 13 to get 1638. Last but not least, the fraction bar has the lowest precedence (5th rank), resulting in 1638 being divided by 2 to obtain 819.

For Example :

0-3-14

On Page 665 you will learn the formula for the sum of a geometric sequence. Don't worry if you do not yet know what a geometric series is. In any case, the formula is

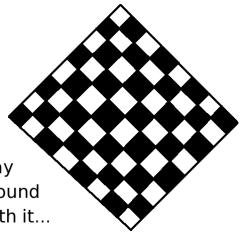
$$S = \frac{a}{1 - c_r}$$

and if you were told $a = 2$ and $c_r = 1/3$ then we would have

$$S = \frac{2}{1 - 1/3}$$

Since the fraction bar is the lowest precedence, we do the subtraction, which is the only remaining operation. This is $1 - 1/3 = 2/3$. Then we have

$$S = \frac{2}{2/3} = 2 \times \frac{3}{2} = 3$$



0-3-15

Try the formula from the previous box with:

- What if $a = 20$ and $c_r = 1/5$? [Answer: $S = 25$.]
- What if $a = 4$ and $c_r = -1/3$? [Answer: $S = 3$.]

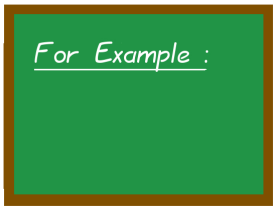
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As I said, we will study the concept of a geometric series on Page 665, so there's no reason for you to know what it is right now. However, if you are burning with curiosity, then you might be happy to know that you've just calculated:

$$\begin{aligned} 2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\cdots &= 3 \\ 20+4+\frac{4}{5}+\frac{4}{25}+\frac{4}{125}+\frac{4}{625}+\cdots &= 25 \\ 4-\frac{4}{3}+\frac{4}{9}-\frac{4}{27}+\frac{4}{81}+\cdots &= 3 \end{aligned}$$

Where the dots are meant to indicate a sum of infinitely many terms.



0-3-16

There is this equation called Fisher's Equation which comes in two forms. (You'll learn about this equation on Page 470.) One form is

$$(1+r_{real})\,(1+r_{infl}) = (1+r_{noml})$$

and another form is

$$r_{real} = \frac{r_{noml} - r_{infl}}{1+r_{infl}}$$

These equations are very important for working with inflation. We're going to use the second form of Fisher's Equation now. Suppose you are asked to find r_{real}, called "the real rate of return," given that $r_{infl} = 0.03$ (the rate of inflation) and $r_{noml} = 0.09$ (the nominal rate). If you are curious, you might want to know that we're computing how fast an investment (that claims to return 9% percent per year) will grow, in terms of its genuine purchasing power, as compared to the amount printed on the bank statement, in the presence of a very typical rate of inflation, namely 3%.

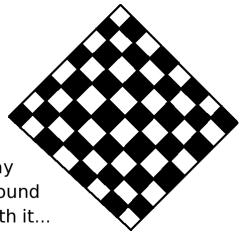
We start with

$$r_{real} = \frac{0.09 - 0.03}{1+0.03}$$

and realize that the fraction bar has lowest precedence (5th rank), so we do that last. We now notice that a subtraction and an addition are present, and those have equal precedence (4th rank). We calculate $0.09 - 0.03 = 0.06$ as well as $1 + 0.03 = 1.03$. Now we have

$$r_{real} = \frac{0.06}{1.03} = 0.0582524\cdots$$

which is the answer to the question. However, if you are curious, that means that the purchasing power would grow at a rate of 5.82% per year.

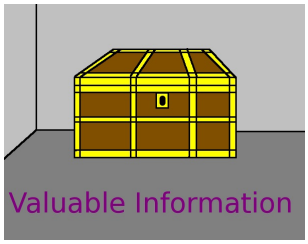


0-3-17

Now let's explore how that answer would change in relationship to what we plugged in. (A mathematician would say, "let's explore how the outputs change depending upon the inputs.")

- What if $r_{infl} = 0.04$, but $r_{noml} = 0.09$ as before?
  [Answer: $r_{real} = 0.0480769\cdots = 4.80\%$.]
- What if $r_{infl} = 0.03$ as the original, but now $r_{noml} = 0.08$?
  [Answer: $r_{real} = 0.0485436\cdots = 4.85\%$.]
- What if $r_{infl} = 0.04$ and $r_{noml} = 0.08$?
  [Answer: $r_{real} = 0.0384615\cdots = 3.84\%$.]

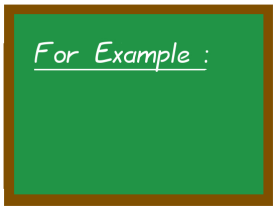
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You might not have seen symbols like r_{infl}, r_{real}, and r_{noml} before. Each one is a "rate," specifically: the real rate, the nominal rate, and the rate of inflation. Therefore, they each tend to be abbreviated as r. The tiny writing under the r is called a *subscript*.

Some books will write r_1, r_2, and r_3, but this is not the best way. That's because it can be hard to remember if r_2 is the real rate, rate of inflation, or the nominal rate. By putting the four-letter subscripts in place of the subscripts "1," "2," and "3," we help reduce the risk of accidentally putting the wrong data in a particular spot.

However, at other times, we will write x_1, x_2, x_3, and x_4 if a problem happens to have four variables. You will find that many economics, science, and finance problems have a surprisingly large number of variables.



0-3-18

Of all formulas in the entire book, the compound interest formula is among the ones that we will use most frequently. We will study it in detail, starting on Page 278. The formula is written

$$A = P(1+i)^n$$

though some textbooks write it in a different form. Let's evaluate this formula to find A where $r = 0.06$, $P = 2400$, $t = 5$, $m = 12$, $i = r/m$ and $n = mt$.

The first thing that might surprise you is that we're given data for t and for r, which are not given in the formula; furthermore, we are not given data for i and for n, which are given in the formula. This happens a lot in finance—and there are good reasons for it too. In any case, we can very quickly use the "helper formulas" $i = r/m$ and $n = mt$ by computing

$$i = \frac{r}{m} = \frac{0.06}{12} = 0.005 \text{ and } n = mt = (12)(5) = 60$$

leaving us with

$$A = 2400 \cdot (1 + 0.005)^{60}$$

which we will solve in the next box.

Continuing with the example of the previous box,

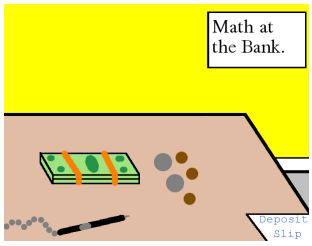
$$\begin{aligned} A &= 2400 \cdot (1 + 0.005)^{60} \\ A &= 2400 \cdot (1.005)^{60} \\ A &= 2400 \cdot 1.34885\cdots \\ A &= 3237.24\cdots \end{aligned}$$

Therefore, we learn that we would have \$ 3237.24 in our account if we were to invest \$ 2400 at 6% compounded monthly for five years. Observe that we did not go step-by-step telling you to do multiplications before additions, and so forth. The time has come for you to understand yourself which operations should be done before which other operations. Nonetheless, we will still show every last step.

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It is perhaps likely that you have never seen a formula broken up with little "helper formulas" before. Accordingly, you might be surprised to see it now. This is actually quite a common practice. Of course, we could have written

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$



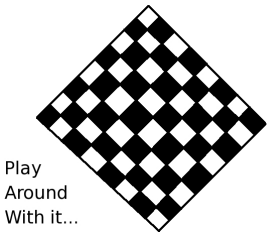
instead of

$$A = P(1+i)^n \text{ where } n = mt \text{ and } i = r/m$$

and that's not so bad of an equation. In fact, many books write exactly that. By plugging the helper formulas into the original, we are left with a larger, more complicated (but not too complicated) equation.

The problem is that certain formulas later in finance would become absolutely enormous if we were not use helper formulas to keep them tidy. Those formulas, which you will see in Chapter Four, would become very hard to memorize, accident prone, and rather hard to read. Therefore, while the idea of helper formulas perhaps is new to you, it is in the end a tremendous reduction in complexity.

It is for this reason that nearly all advanced finance books, and this textbook also, use helper formulas. Sometimes these are called auxiliary formulas.



0-3-19

Now let's use the formula $A = P(1+i)^n$ with the following data:

- Use $r = 0.03$, $P = 2400$, $t = 5$, $m = 12$, $i = r/m$ and $n = mt$. [Answer: \$ 2787.88.]
- Use $r = 0.12$, $P = 2400$, $t = 5$, $m = 12$, $i = r/m$ and $n = mt$. [Answer: \$ 4360.07.]
- Use $r = 0.2995$, $P = 2400$, $t = 5$, $m = 12$, $i = r/m$ and $n = mt$. [Answer: \$ 10,533.77.]

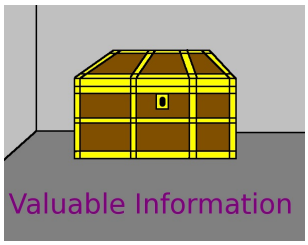


As you can see from the previous example, and the first two bullets of the previous box, with interest rates 3%, 6%, and 12% we obtained the amounts \$ 2787.88, \$ 3237.24, and \$ 4360.07. Therefore, if you double the interest rate, you do not necessarily double the amount.

Nonetheless, many students imagine that if you do double the interest rate that you will double the amount.

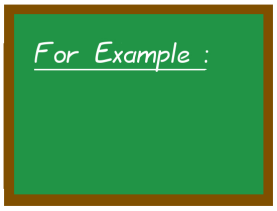
This is an example of how you must not allow intuition to get in the way of computation. A human's intuition is often wrong, but computation in the hands of a person sufficiently skilled is rarely wrong.

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Just a note: it is important not to round off interest rates. When computing with interest (but also in any other computation) it is a great idea to try to keep the numbers on your calculator's screen. You should not copy them down onto a piece of paper and re-enter them.

First, you will save both the time of writing and the time of rekeying in the number (and on a quiz or exam, time can matter a lot.) Second, you can make a key-press error or a writing error if you transfer numbers between the page and the calculator. Third, you probably won't find yourself writing down all 10 or so digits of precision that your calculator has displayed for you, and so you will introduce rounding error. Fourth, and most mathematics faculty do not know this, your calculator usually has additional digits of precision that it is not displaying to you. If you keep the numbers on screen and in memory, those digits are always at work. However, if you copy and retype, then that extra precision is lost.



0-3-20

The formula we're about to use is called "the future value" formula and is used when individuals save up for retirement, buying a house, or any other reason. It is written

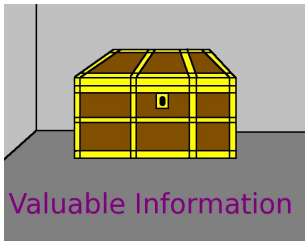
$$FV = c \cdot \frac{(1+i)^n - 1}{i}$$

and we will study this formula on Page 641. Now, we will now compute using this formula with the data $c = 250$, $r = 0.07$, $t = 30$, $m = 12$, $i = r/m$, $n = mt$.

Using the helper formulas we learn $i = 0.005833\overline{3}$ and $n = 360$. Then we have

$$\begin{aligned} FV &= 250 \cdot \frac{(1+0.005833\overline{3})^{360} - 1}{0.005833\overline{3}} \\ FV &= 250 \cdot \frac{(1.005833\overline{3})^{360} - 1}{0.005833\overline{3}} \\ FV &= 250 \cdot \frac{8.11649\cdots - 1}{0.005833\overline{3}} \\ FV &= 250 \cdot \frac{7.11649\cdots}{0.005833\overline{3}} \\ FV &= 250 \cdot 1219.97\cdots \\ FV &= 304,992.\cdots \end{aligned}$$

If you are curious, this means that if you make monthly deposits of \$ 250 for 30 years, while earning 7% interest, then at the end you will have \$ 304,992. Now is also a good time to remind you that we are not at all trying to memorize these formulas—we are only trying to use them.



Sometimes students don't know how to do large exponents on their calculator. This is often because the preceding algebra course dealt mainly with squares and cubes.

There can be a lot of variation from calculator to calculator, but usually the key either looks like a hat: $\hat{}$ or like x^y but maybe y^x.

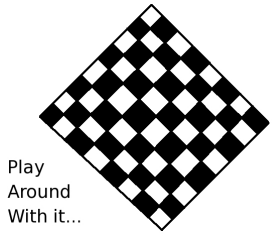
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I'd like to mention, really quickly, that FV is one symbol, representing "future value." It does not represent F times V.

This is a bit unfortunate, as it can confuse students, but the fact of the matter is that there are just too many quantities to represent in finance and economics, and not enough letters of the alphabet.

Therefore, we are compelled to use pairs of letters once in a while.



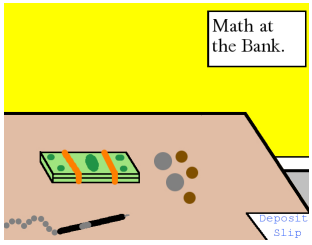
0-3-21

Try it yourself with the following data.

- If $r = 0.07$, $c = 100$, $t = 35$, $m = 52$, $n = mt$, $i = r/m$, what is FV? [Answer: \$ 785,145.76.]
- If $r = 0.07$, $c = 148$, $t = 30$, $m = 52$, $n = mt$, $i = r/m$, what is FV? [Answer: \$ 786,602.20.]
- If $r = 0.07$, $c = 223$, $t = 25$, $m = 52$, $n = mt$, $i = r/m$, what is FV? [Answer: \$ 786,512.69.]
- If $r = 0.07$, $c = 347$, $t = 20$, $m = 52$, $n = mt$, $i = r/m$, what is FV? [Answer: \$ 786,559.59.]

Remember, your answer should match mine to five significant figures, if not six!

As we noted before, it is important to keep as many digits of precision as possible. The painless way to do that is to keep the numbers on the calculator's screen, continuously, and let it do the work for you. Take care never to round off an interest rate.

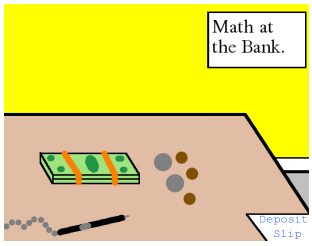


You're probably wondering what you just calculated two boxes ago. Imagine four brothers, Ned, Ted, Ed, and Fred. Each of them has access to a mutual fund that reliably pays 7%, compounded weekly. Each of them also wants to retire at age 65.

- Ned is very prudent and begins saving for retirement at age 30. This means he has 35 years to save up, and that's why $t = 35$ for him. He deposits \$ 100 per week. That's an extremely small sacrifice for a mid-career professional to make, and it is heart-warming to see that he can retire with three-quarters of a million dollars.
- Ted is just a little less prudent and begins saving for retirement at age 35. This means he has 30 years to save up, and that's why $t = 30$ for him. As it turns out, he must deposit \$ 148 per week to achieve the same goal as Ned. Well, that's not so bad.

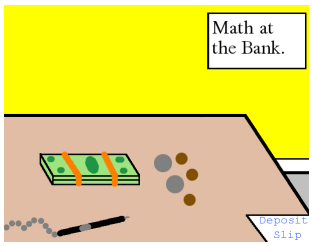
We will continue this discussion in the next box.

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Continuing with the previous box. . . ,

- Ed was still less of a planner and didn't begin saving for retirement until he was age 40. This means he has 25 years to save up, which is still a respectable amount of time. However, Ed pays an extremely stiff price for his delay. Ed must deposit \$ 223, *which is more than double what Ned is paying*, in order to achieve the same goal as Ned.
- Fred has been very profligate, and does not start saving for retirement until he is 45 years old. This means he has "only" twenty years to save up. However, the consequences to Fred's lifestyle will be very serious. To achieve the same goal as Ned, it will be necessary that Fred deposit \$ 347 per week—a rather large deposit. That's *more than triple* what Ned is depositing and *more than double* what Ted is depositing.

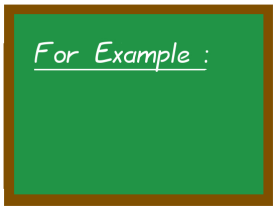


Continuing with the analysis of the last two boxes, is it not shocking that the consequences for Fred are so grave? After all, isn't 20 years (Fred's timeline) more than half of 35 years (Ned's timeline)?

Now, I hope you understand why I wrote this book. Financial mathematics and mathematical economics are *very* complicated. Small misunderstandings can have drastic life consequences. Human intuition is often shocked when presented with cold facts of precise computation.

One last note is that we did this extended calculation for Ned, Ted, Ed, and Fred without adjusting for inflation. We should adjust for inflation, and we'll learn how to do that on Page 661.

The following model is quoted from *Calculus—Early Transcendentals* by Prof. Soo Tan, first edition, Chapter 6.4, Example 6.



Some interesting models occur in environmental science. Suppose that the oxygen content t days after organic waste has been dumped into a pond is given by

$$f(t) = 100\left(\frac{t^2+10t+100}{t^2+20t+100}\right)$$

percent of its normal level. Let's compute $f(20)$.

The computation is in the next box.

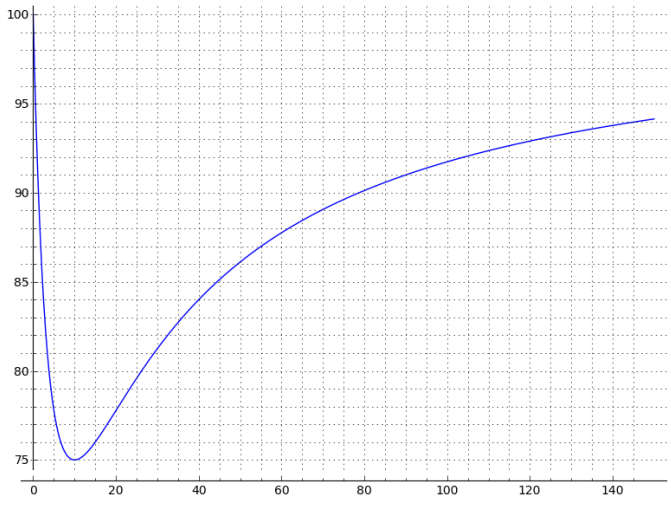
0-3-22

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Continuing with the previous box...

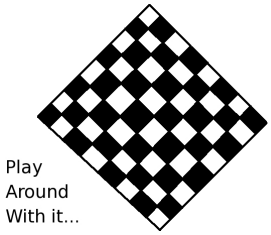
$$\begin{aligned} f(t) &= 100\left(\frac{(20)^2+10(20)+100}{(20)^2+20(20)+100}\right) \\ &= 100\left(\frac{400+10(20)+100}{400+20(20)+100}\right) \\ &= 100\left(\frac{400+200+100}{400+400+100}\right) \\ &= 100\left(\frac{700}{900}\right) \\ &= 100(0.7\overline{7}) \\ &= 77.7\overline{7} \end{aligned}$$

We see that the oxygen level will be 77.77% of normal at that point, which is probably enough to kill more than a few fish. That's why toxic waste dumping is illegal.



This is a graph of Prof. Tan's model of the oxygen content of the lake, from the previous example. At the very first moment after a bunch of organic waste is dumped into the lake, the lake hasn't had any time to realize that anything has happened, so the graph starts at 100%. Then fairly rapidly, bacteria will notice all the yummy organic waste and feast on it, and reproduce. All of those new and old bacteria will consume a great deal of the lake's oxygen, and relatively quickly the oxygen level drops.

Eventually, most of the waste is consumed, and the bacteria will return to a more normal population level, and the oxygen level in the lake will begin to move toward the usual levels, as nature restores its balance. However, it will take a long time to completely recover, as even 120 days out we can see that the oxygen level is only at about 93% of normal.

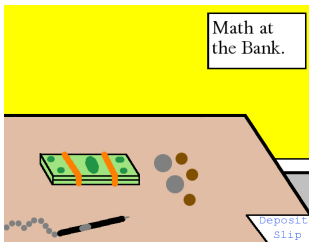


0-3-23

Now try it yourself with the following values.

- What about $t = 40$? [Answer: 84%.]
- What about $t = 60$? [Answer: $87.7551\cdots$%.]
- What about $t = 80$? [Answer: $90.1234\cdots$%.]
- If you got the first three correct, you can skip the last two.
- What about $t = 100$? [Answer: $91.7355\cdots$%.]
- What about $t = 120$? [Answer: $92.8994\cdots$%.]

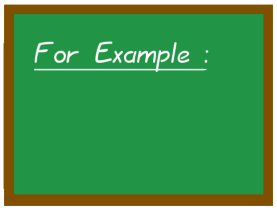
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Quality control is a very important issue in modern manufacturing processes. If you have n devices, and each has a probability q of being defective, then the probability that you will have exactly 2 out of n defective devices is given by Bernoulli's formula

$$p = \frac{n(n-1)}{2} \cdot q^2(1-q)^{n-2}$$

but don't worry if you are totally unfamiliar with the idea of probability. We will study this later, on Page 1014.



0-3-24

For now, let's practice with Bernoulli's formula where $q = 0.01$, representing a defective rate of 1%, and a batch of $n = 500$ video screens for smart phones.
We would obtain

$$\begin{aligned} p &= \frac{500(500-1)}{2} \cdot (0.01)^2(1-0.01)^{500-2} \\ p &= \frac{500(499)}{2} \cdot (0.01)^2(0.99)^{498} \\ p &= \frac{500(499)}{2} \cdot (0.0001)(0.00670389\cdots) \\ p &= \frac{249,500}{2} \cdot (0.000000670389\cdots) \\ p &= 124,750 \cdot (0.000000670389\cdots) \\ p &= 0.0836310\cdots \end{aligned}$$

Therefore, we learn there is roughly a probability of $0.0836310\cdots$ (or a 8.36% chance) that exactly 2 of the smart-phone video screens will be defective.



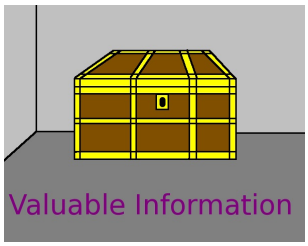
Many students, upon seeing the intermediate value of

$$0.000000670389\cdots$$

that we had in the previous box, would simply treat this number as zero.

However, if you were to have done that, then you'd have said that there was 0 probability, or a 0% chance, of there being exactly two defective video screens. This is clearly a gross misestimation!

Therefore, you should not arbitrarily decide that a number, in the middle of a calculation, is "close enough to zero." At the end of a calculation, perhaps, then you can use some judgement, because you have some idea of what the number is trying to tell you. For intermediate steps, however, it is extremely dangerous to round.



On a quiz or exam, if you're running low on time, then perhaps you don't want to write all the zeros in

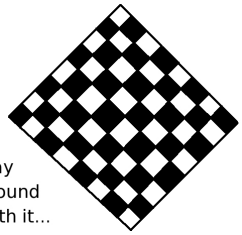
$$0.000000670389\cdots$$

especially because you might have to count carefully to make sure you put the right number of zeros. The solution to that dilemma is to use scientific notation, and write

$$6.70389\cdots \times 10^{-7}$$

which takes less time. Furthermore, your calculator probably has a button to make that conversion for you—and that would save you the time of counting out the seven zeros.

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0-3-25

Using Bernoulli's Formula

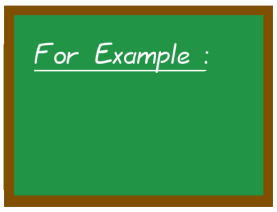
$$p = \frac{n(n-1)}{2} \cdot q^2(1-q)^{n-2}$$

what is the probability p, that among $n = 200$ keyboards, I have exactly two that are defective...

- ...with defective rate q of two percent? [Answer: $p = 0.145772\cdots$.]
- ...with defective rate q of one percent? [Answer: $p = 0.272033\cdots$.]
- ...with defective rate q of half a percent? [Answer: $p = 0.184400\cdots$.]

Sorry to remind you twice, but your answer should match mine to five significant figures.

Now would be a good time to point out that "exactly two" is just one version of Bernoulli's formula. You'll learn later, on Page 1014, how to use it when you want to know "exactly one," or "exactly three," and even "two or more."



0-3-26

The Quadratic Equation is an extremely useful formula. Probably you've seen it before, but perhaps not very recently. Here it is:

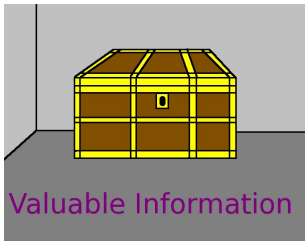
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and it is used to solve equations involving a second-degree polynomial. If we want to solve

$$4x^2 - 144x + 1292 = 0$$

then we would have $a = 4$, $b = -144$, and $c = 1292$.

A brief note about the $\pm$ symbol is in order before we continue.



Note: we have not talked about the symbol $\pm$ before in this textbook. This symbol exists because there are two values of x that the formula is going to deliver to you. One of those values of x is found by using the $\pm$ as a plus sign, and the other value of x is found by using the $\pm$ as a minus sign.

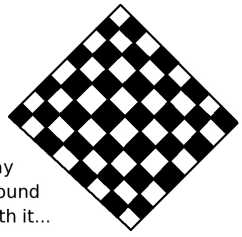
We will now use the quadratic equation in the next box to solve the example problem in the previous box.

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Continuing with the previous box,

$$\begin{aligned} x &= \frac{-(-144) \pm \sqrt{(-144)^2 - 4(4)(1292)}}{2(4)} \\ &= \frac{144 \pm \sqrt{20,736 - 4(4)(1292)}}{2(4)} \\ &= \frac{144 \pm \sqrt{20,736 - 20,672}}{8} \\ &= \frac{144 \pm \sqrt{64}}{8} \\ &= \frac{144 \pm 8}{8} \\ &= \frac{144 + 8}{8} \text{ or } \frac{144 - 8}{8} \\ &= \frac{152}{8} \text{ or } \frac{136}{8} \\ &= 19 \text{ or } 17 \end{aligned}$$

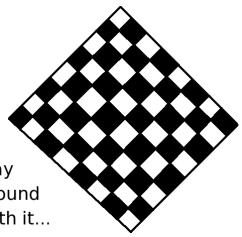
Now, you can plug $x = 19$ and $x = 17$ back into the original quadratic equation $4x^2 - 144x + 1292 = 0$ to verify that they are actually correct.



0-3-27

Try it yourself with the following data.

- Solve $2x^2 - 140x + 2418 = 0$. [Answer: $x = 39$ and $x = 31$.]
- Note: if you're having trouble with these, we have an entire module "Solving Quadratic Equations," which begins on Page 103.
- Solve $-3x^2 - 36.9x + 90.72$. [Answer: $x = 2.1$ and $x = -14.4$.]
- Solve $-4x^2 - 405.6x - 9897.8 = 0$. [Answer: $x = -60.5$ and $x = -40.9$.]
- If you got the first three correct, you can move on to the next box now.
- Solve $6x^2 + 355.2x - 18105.6 = 0$. [Answer: $x = 32.8$ and $x = -92$.]



0-3-28

The following two quadratic equation problems are intended to demonstrate to you how sensitive the quadratic equation can become to rounding error.

- Solve $x^2 + 6.001x + 9 = 0$. [Answer: $x = -2.94572\cdots$ or $x = -3.05527\cdots$.]
- The next one is so sensitive to rounding error, it will be necessary that you use nine significant digits, rather than six, to make it come out correctly.
- Solve $0.1x^2 + 85x + 0.2 = 0$. [Answer: $x = -0.00235294768\cdots$ or $x = -849.997647\cdots$]

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For Example :

0-3-29

The equation that governs car loans, mortgages, and related financial instruments is

$$PV = c \cdot \frac{1-(1+i)^{-n}}{i}$$

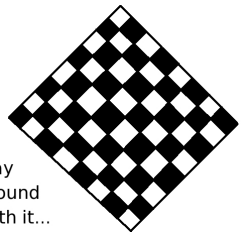
with $i = r/m$ and $n = mt$. We'll consider a 30-year mortgage ($t = 30$), with monthly ($m = 12$) payments equaling \$ 760 ($c = 760$), at an interest rate of 4.25%, ($r = 0.0425$). The PV will be the amount of the bank's money you are permitted to spend in buying the house (to which you add the down payment from your own money.) Let's calculate the PV.

We begin by finding $n = mt = (12)(30) = 360$ and $i = 0.0425/12 = 0.00354166\overline{6}$.

$$\begin{aligned}
PV &= 760 \cdot \frac{1-(1+0.00354166\overline{6})^{-360}}{0.00354166\overline{6}} \\
PV &= 760 \cdot \frac{1-(1.00354166\overline{6})^{-360}}{0.00354166\overline{6}} \\
PV &= 760 \cdot \frac{1-0.280061\cdots}{0.00354166\overline{6}} \\
PV &= 760 \cdot \frac{0.719938\cdots}{0.00354166\overline{6}} \\
PV &= (760\cdots)(203.276\cdots) \\
PV &= 154,490.\cdots
\end{aligned}$$

Therefore, the bank will be giving \$ 154,490 with which to buy the house. If the homeowner perhaps puts a down payment of \$ 18,000, then she or he can buy a house with the asking price of \$ 172,490. We will study these matters in detail later, starting on Page 579.

Perhaps you might not have seen a negative exponent before—that's okay. We'll discuss that in more detail, in the "How Exponents Work" module, starting on Page 338. However, if you are curious now, you can go ahead and read that module at this time. It does not depend on any of the modules between this page and the start of that module.



0-3-30

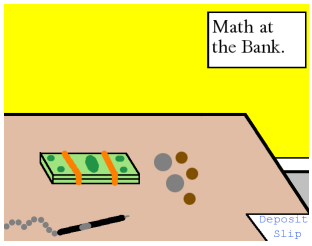
We're going to practice the PV equation with $c = 800$, $n = 360$, $m = 12$ and r varying as listed below. Remember that $i = r/m$, and

$$PV = c \cdot \frac{1-(1+i)^{-n}}{i}$$

- If $r = 0.04$ then what is PV? [Answer: $PV =$ \$ 167,568.99.]
- If $r = 0.06$ then what is PV? [Answer: $PV =$ \$ 133,433.29.]
- If $r = 0.08$ then what is PV? [Answer: $PV =$ \$ 109,026.79.]
- If you've gotten the first three correct, you can skip the last two.
- If $r = 0.05$ then what is PV? [Answer: $PV =$ \$ 149,025.29.]
- If $r = 0.07$ then what is PV? [Answer: $PV =$ \$ 120,246.05.]

Your answer should match mine to five significant figures. (This is the last reminder, I promise!)

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The calculation in the previous box represents making an $ 800 dollar mortgage payment each month for a standard 30-year mortgage. As you can see, tiny changes in the interest rate (only 1%) made very significant changes in the PV. The value of the PV is how much of a condominium or house the bank is willing to let you buy.

The moral of the story is that small changes in the interest rate are a very big deal for mortgages. This is why you sometimes hear news stories about "the Fed" thinking about possibly raising interest rates by 1/4th of 1%. That rate which the Fed sets will affect the mortgage interest rate that new homebuyers can expect.



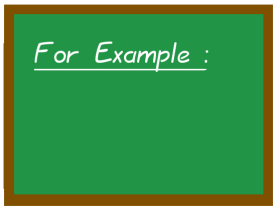
Okay, do you remember how back on Page 68, I had said that formulas in finance can become huge if we reject the use of helper or auxiliary formulas? Let's contemplate the formula we just used. While

$$PV = c \cdot \frac{1-(1+i)^{-n}}{i}$$

is not the simplest formula, certainly

$$PV = c \cdot \frac{1-(1+\frac{r}{m})^{-mt}}{\frac{r}{m}}$$

is much worse, don't you think? Other formulas, in later finance courses, would be still uglier without the use of helper or auxilliary formulas.



0-3-31

The hardest problem on Diagnostic One, at least from my perspective, involves the following pair of equations. Surely they look a little strange, and we'll study them more on Page 1169, but for now, let's just see if we can use them.

When designing a soda can, an oil drum, or any cylinder, the minimal amount of metal should be used to keep costs down, which means minimizing the surface area. For any volume V, that minimum will be achieved by radius r and height h where

$$r = \sqrt[3]{\frac{V}{2\pi}} \text{ and } h = \sqrt[3]{\frac{4V}{\pi}}$$

remembering that $\pi \approx 3.14159265358979323846264\cdots$.

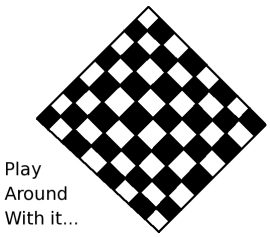
Now, let's use these to find the radius and height of the optimal one gallon can. A quick internet search reveals that a gallon is $V = 3785.41\cdots$ cubic centimeters. We will solve this problem in the next box.

Continuing with the previous box,

$$\begin{array}{rcl} r &=& \sqrt[3]{\frac{3785.41\cdots}{2(3.14159\cdots)}} \\ r &=& \sqrt[3]{\frac{3785.41\cdots}{6.28318\cdots}} \\ r &=& \sqrt[3]{602.466\cdots} \\ r &=& 8.44587\cdots \end{array} \qquad \begin{array}{rcl} h &=& \sqrt[3]{\frac{4(3785.41\cdots)}{3.14159\cdots}} \\ h &=& \sqrt[3]{\frac{15{,}141.6\cdots}{3.14159\cdots}} \\ h &=& \sqrt[3]{4819.73\cdots} \\ h &=& 16.8917\cdots \end{array}$$

Therefore, our final answer is that the radius should be $8.44587\cdots$ cm and the height should be $16.8917\cdots$ cm. Does this look familiar? It should! It was the middle bullet of the checkerboard on Page 63 where you practiced the formula for the volume of a cylinder. On that page, we found that the volume is 3785.40, and so we know at least that part of the problem is correct.

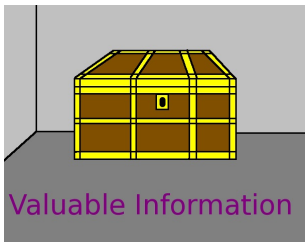
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0-3-32

For trying the cylinder problem yourself, I think it suffices to just consider an objective of designing a can with 1000 cubic centimeters of volume. Using the formulas from two boxes ago...

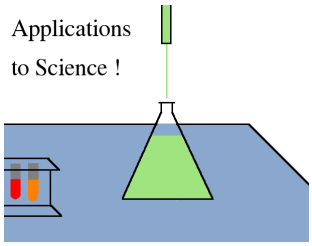
- ...what is the optimal radius? [Answer: $5.41926\cdots$ cm.]
- ...what is the optimal height? [Answer: $10.8385\cdots$ cm.]
- Using the formula for the volume of a cylinder, $V = \pi r^2 h$, what volume do you expect? [Answer: $999.997\cdots$, which is close enough.]



Some calculators have a cube-root key, and most do not. If it does exist, then it would look like $\sqrt[3]{\ }$.

However, if not, then there is a key for arbitrary roots. It usually looks like $\sqrt[x]{y}$ or like $\sqrt[y]{x}$. That way, you can take a cube root, a fourth root, a fifth root, a sixth root, or whatever you need.

You can check the manual of your calculator for further details.



Some students might imagine that I'm using a rarely-need tool when I discuss cube roots, or especially higher roots like 4th roots, 5th roots, and so forth.

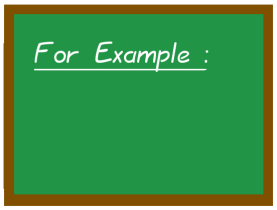
Outside of finance, for example in science and other applied subjects, the higher roots are uncommon. On the one hand, the square root comes up in too many situations to list, and the cube root comes up frequently in problems relating to volumes or three-dimensional figures. On the other hand, the fifth, sixth, and further roots do not occur very often at all, but a notable exception is the important role of the fourth root in radar. We'll also see the 3/2th root and the 2/3rds root in relation to Kepler's Laws of planetary motion.

Surprisingly, in finance, the higher roots are indeed very important. For example, we'll require a 5th root on Page 407, a 44th root on Page 462, a 12th root on Page 424, and a 872nd root on Page 351, as well as a 4th root on Page 544. You can flip there and check if you don't believe me.

At this point, we have completed our practice with plugging numbers into formulas. If you've successfully reached this point, getting the correct answers along the way, then you are very ready for Diagnostic Two.

However, I have some extremely handy tips and tricks that I'd like to share with you. Also, I'd like to warn you of a family of incredibly common mistakes that many students make, even though each is quite avoidable. We'll discuss those traps first, and then I'll share with you my four handy tricks.

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0-3-33

Let's say you are presented with the following question

$$\text{What is } 16^2 + 5^2?$$

as well as

$$\text{What is } (16 + 5)^2?$$

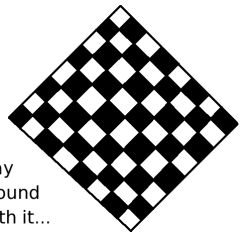
Now in the first case, there are no parentheses (1st rank) and so we go to exponents (2nd rank). The exponents call for squaring the 16 and the 5, and so we have $256 + 25$. This is just an addition, and comes to 281.

However, in the second case, we do have parentheses (1st rank), thus we must first add the 16 and the 5, and we get 21. Then we can square the 21 to get 441.

Does $281 = 441$? No, of course not. However, this means in general, the rule

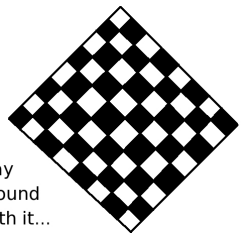
$$x^2 + y^2 = (x + y)^2 \qquad \leftarrow \text{WRONG!}$$

is not true. You cannot treat $(x + y)^2$ and $x^2 + y^2$ as synonyms. They are different. Nonetheless, students will treat them as the same, and it is an extremely common error.



0-3-34

- What is $(9 + 4)^2$? [Answer: 169.]
- What is $9^2 + 4^2$? [Answer: 97.]
- Are your answers to the first two questions the same? or different? [Answer: "they are different."]



0-3-35

- What is $(5 + 3)^3$? [Answer: 512.]
- What is $5^3 + 3^3$? [Answer: 152.]
- Are your answers to the first two questions the same? or different? [Answer: "they are different."]

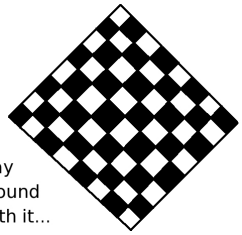


Just like before, this means in general, the rule

$$x^3 + y^3 = (x + y)^3 \qquad \leftarrow \text{WRONG!}$$

is not true. You cannot treat $(x + y)^3$ and $x^3 + y^3$ as synonyms. They are different. Nonetheless, students will treat them as the same, and lose points. This mistake is less common than the one with squares, but it is still common.

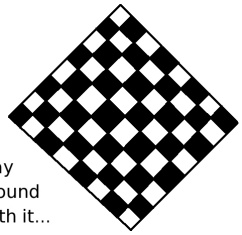
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0-3-36

Evaluate the following when $x = -1$ and $y = 2$.

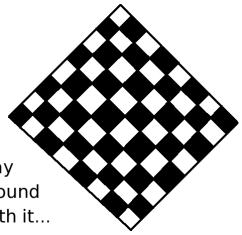
- What is $\sqrt{x^2 + y^2}$? [Answer: $2.23606\cdots$.]
- What is $\sqrt{x^2} + \sqrt{y^2}$? [Answer: 3.]
- Are your answers to the first two questions the same? or different? [Answer: "they are different."]
- Therefore, is it the case that $\sqrt{x^2 + y^2}$ and $\sqrt{x^2} + \sqrt{y^2}$ are synonyms? [Answer: No.]



0-3-37

Evaluate the following when $x = 25$ and $y = 144$.

- What is $\sqrt{x + y}$? [Answer: 13.]
- What is $\sqrt{x} + \sqrt{y}$? [Answer: 17.]
- Are your answers to the first two questions the same? or different? [Answer: "they are different."]
- Therefore, is it the case that $\sqrt{x + y}$ and $\sqrt{x} + \sqrt{y}$ are synonyms? [Answer: No.]



0-3-38

Evaluate the following when $x = -3$ and $y = 4$.

- What is $\sqrt{x^2} + \sqrt{y^2}$? [Answer: 7.]
- What is $x + y$? [Answer: 1.]
- Are your answers to the first two questions the same? or different? [Answer: "they are different."]
- Therefore, is it the case that $\sqrt{x^2} + \sqrt{y^2}$ and $x + y$ are synonyms? [Answer: No.]



By now you have surely become bored with this pattern, but we have learned that the rule

$$\sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2} \qquad \leftarrow \text{WRONG!}$$

is not true. You cannot treat $\sqrt{x^2 + y^2}$ and $\sqrt{x^2} + \sqrt{y^2}$ as synonyms. They are different.

This particular error is extremely common. I think part of the reason is that students further imagine

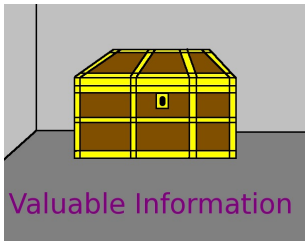
$$\sqrt{x^2} + \sqrt{y^2} = x + y \qquad \leftarrow \text{WRONG!}$$

which is also wrong, as we showed in the previous box.

These two steps (both incorrect) would turn the rather nasty $\sqrt{x^2 + y^2}$ in to the far more ordinary $x + y$. As a result, this makes the error extremely attractive.

No matter how convenient it would be if these steps were legitimate, the fact remains that they are not valid, as we've seen.

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In summary, we have just illuminated five major algebraic misconceptions:

$$\begin{array}{lrcll} (1) & x^2+y^2 & = & (x+y)^2 & \leftarrow \text{WRONG!} \\ (2) & x^3+y^3 & = & (x+y)^3 & \leftarrow \text{WRONG!} \\ (3) & \sqrt{x^2+y^2} & = & \sqrt{x^2}+\sqrt{y^2} & \leftarrow \text{WRONG!} \\ (4) & \sqrt{x+y} & = & \sqrt{x}+\sqrt{y} & \leftarrow \text{WRONG!} \\ (5) & \sqrt{x^2}+\sqrt{y^2} & = & x+y & \leftarrow \text{WRONG!} \end{array}$$

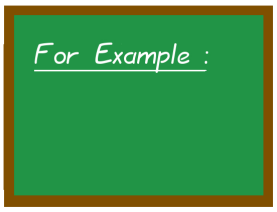


Now, there can be particular values of x and y where any of those statements are true, such as if either $x = 0$ and $y = 0$, but that does not matter.

In order for a rule to be useable in some setting, it must be true for all possible x that could occur, and all possible y that could occur.

Furthermore, by producing counterexamples, we have convinced ourselves that these rules are not true.

That concludes our discussion of common pitfalls. Now, for something completely different. While we are on the topic of calculator use, I'm going to share with you a few useful calculator tricks.



0-3-39

Suppose you are asked the following question: "Please write the following numbers on the line below, sorted from smallest to largest."

$$\pi/7, \quad 9/17, \quad 1, \quad 0.62, \quad 7/5, \quad (7/9)^3, \quad \frac{1}{\sqrt{6}}$$

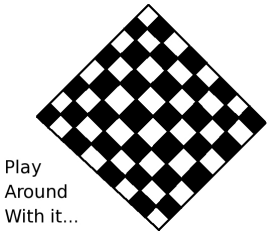
The key here is to remember that calculators tend to convert numbers into decimal notation. In that way, decimal notation is not unlike Grand Central Station, in that lots and lots of trains go there, and so it is a great place to change trains (or change from one notation to another.) Let's convert all those numbers to decimal notation:

$$\begin{array}{lcl} \pi/7 & \to & 0.448798\cdots \\ 9/17 & \to & 0.529411\cdots \\ 1 & \to & 1.00000\cdots \\ 0.62 & \to & 0.620000\cdots \end{array} \qquad \begin{array}{lcl} 7/5 & \to & 1.40000\cdots \\ (7/9)^3 & \to & 0.470507\cdots \\ \frac{1}{\sqrt{6}} & \to & 0.408248\cdots \end{array}$$

Now the ordering is obvious. An easy way to carry out the ordering is to start with the smallest number, and cross off the decimal as you put it into the ordering. The final ordering is

$$\frac{1}{\sqrt{6}}, \quad \pi/7, \quad (7/9)^3, \quad 9/17, \quad 0.62, \quad 1, \quad 7/5$$

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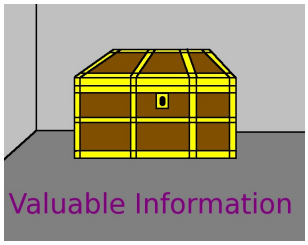


0-3-40

Now try it yourself, with these numbers:

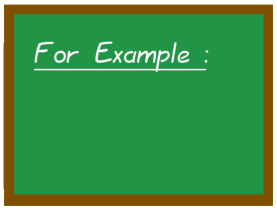
$$\pi/5, \quad (8/9)^3, \quad 11/17, \quad 1, \quad 6/5, \quad 0.68, \quad \frac{1}{\sqrt{2}}$$

The answers can be found on Page 85.



We need two common pieces of terminology before we continue.

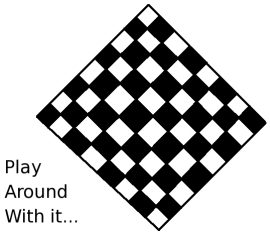
- When we write 3.5 as 7/2, we say that we are writing an *improper fraction*—even though it is the form that mathematicians prefer.
- Likewise when we write 3.5 as $3\frac{1}{2}$, we say this is a *mixed number*, which I guess would be more properly written as $3 + \frac{1}{2}$.



0-3-41

Let's suppose you're asked to convert 173/14 into a mixed number. How would we do that?

1. We need to find the integer part. We use the calculator to find that $173 \div 14 = 12.3571\cdots$. So the integer part is 12.
2. We leave the ugly number $12.3571\cdots$ on the screen, and subtract 12. This is $0.357142\cdots$ and is the fractional part, but written as a decimal.
3. Because we don't want the fractional part as a decimal, and because we want it in 14ths, we multiply $0.357142\cdots$ by 14. Then we learn that the fractional part is 5/14 because the calculator displays "5."
4. We now conclude that the improper fraction 173/14 should be written as the mixed number $12\frac{5}{14}$.



0-3-42

Write the following improper fractions as mixed numbers:

- How about 19/4? [Answer: $4\frac{3}{4}$.]
- How about 19/7? [Answer: $2\frac{5}{7}$.]
- How about 170/13? [Answer: $13\frac{1}{13}$.]

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For Example :

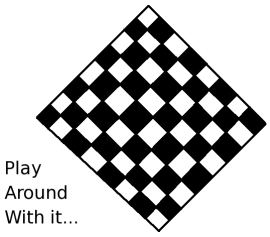
0-3-43

Now, let's see how to reverse the process. This is easier. If someone presents you with $6\frac{7}{12}$, and they want you to convert it from a mixed number to an improper fraction, you would proceed as follows.

1. First, we need to convert 6/1 into something "over 12." So we multiply $6 \times 12 = 72$. Now we know that $6/1 = 72/12$.
2. Second, we realize that

$$6 + \frac{7}{12} = \frac{72}{12} + \frac{7}{12} = \frac{72+7}{12} = \frac{79}{12}$$

The final answer is 79/12. By the way, most students will take the 72 from Step One, and just add the original 7 from the 7/12, and get 79 directly. However, I like to show all the steps. Either way is fine, so long as you get the right answer.



0-3-44

Convert the following mixed numbers into improper fractions:

- How about $6\frac{3}{8}$? [Answer: 51/8.]
- How about $4\frac{2}{3}$? [Answer: 14/3.]
- How about $5\frac{1}{4}$? [Answer: 21/4.]

For Example :

0-3-45

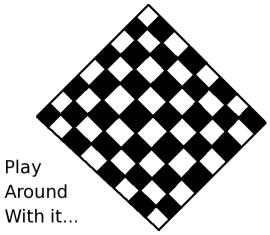
Let's say that you finish a long problem, and get a final answer that something will require $3.69583\overline{3}$ hours. That's not a human readable answer, and certainly not an answer that your boss's boss will understand. The goal is to write this number in the form #h #m #s, representing some number of hours, minutes and seconds.
We start with $3.69583\overline{3}$ on the calculator screen.

1. It is pretty clear we'll have 3 hours, so we subtract 3.
2. Now we have $0.695833\overline{3}$ on screen, representing some fraction of an hour. Multiply it by 60, to find out how many minutes that will be.
3. At this point, we have 41.75 on screen, so we subtract 41, and write down 41 minutes.
4. We have 0.75 on screen, representing some fraction of a minute. We multiply by 60, to find out how many seconds that will be.
5. Finally, we see 45, and so we write down 45 seconds.

Our final answer is 3 hours, 41 minutes, and 45 seconds, which we record as 3h41m45s. If you are sharing this with a boss, you'd probably say 3 hours and 42 minutes, to avoid sounding excessively precise.

Note that 3h41m45s can be written 3hr-41min-45-sec, or 3:41:45, but the most clear of all is to simply say "3 hours, 41 minutes, and 45 seconds."

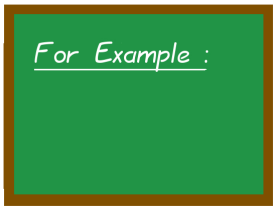
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0-3-46

Convert the following numbers of hours into hours, minutes, and seconds.

- What is $4.3\overline{3}$ in hours, minutes, and seconds? [Answer: 4h20m0s.]
- What is $1.1\overline{1}$ in hours, minutes, and seconds? [Answer: 1h6m40s.]
- What is $5.10194\overline{4}$ in hours, minutes, and seconds? [Answer: 5h6m7s.]



0-3-47

There is a technicality that I forgot to mention, but for completeness, I think I should briefly mention it. Multiple exponentiation goes "top to bottom" as follows:

$$5^{4^{3^2}} = 5^{4^9} = 5^{262,144} = \text{something huge}$$

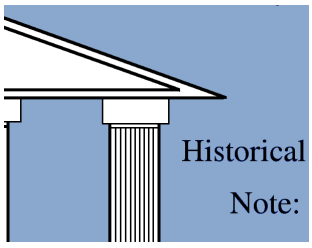
Clearly, this is a very artificial example, and this is even more the case because $5^{262,144}$ turns out to be a 183,231-digit number. You'll learn how to calculate that when you learn about logarithms on Page 504.



The reason that we included the above example is because the following formula is very important in intermediate-to-advanced statistics courses:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and this formula shall govern anything that is random and "Gaussian," such as human height, machining error in manufacturing, human performance on standardized tests, experiments with 1000s or more trials, and many other things. As you can see, there is a multi-level exponent in that formula. The word "Gaussian" comes from Johann Carl Friedrich Gauss (1777–1855) who discovered the formula. You will learn about Gauss later.



This module is over now, but before you close the book, I'd like to share an amusing (and frequently repeated) engineering story with you, even though I have not been able to verify if it actually happened. In the late 19th century and early 20th century, lots of large telescopes were being built at colleges to help students study astronomy. A requisition for a circular lens with 200 square inches of area was requested by an astronomy professor, and the purchasing clerk called the lens supplier. The supplier replied that lenses are ordered by radius, not by area. The purchasing clerk then used 22/7 for π instead of $3.14159\cdots$

The radius, using 22/7 for pi, came to be $7.97724\cdots$ inches. The true value for the radius, calculated with the true value of π is $7.97884\cdots$. The error happens to be $0.00160525\cdots$ inches and that's small. It comes to $40.7734\cdots$ microns. Nonetheless, the discrepancy was enough that the lens did not fit properly into the telescope, and the cost of the lens was deducted from the purchasing clerk's meager salary.

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The reason I find this story hard to believe is that hand-held calculators would not have been available at the time. (Those became available in the 1970s.) Instead, "slide rules" were used for computation. A slide rule is nifty gadget that looks like three rulers glued together such that the middle ruler can move. In any case, each ruler has a number line, and π is almost always marked with a mark of its own. To use $22/7$ would require dividing by 7 and then multiplying by 22, or multiplying by 22 and then dividing by 7—either way, two operations. However, dividing by the π marking would be one operation. Even if the purchasing clerk thought that π and $22/7$ were synonyms, why would the clerk do two steps when he could have done only one?

Sadly, many of the best stories in history turn out to be of doubtful accuracy on closer examination.



Here is the answer to the checkerboard from Page 82 where you were asked to sort a sequence of numbers.

$$\pi/5,\ \ 11/17,\ \ 0.68,\ \ (8/9)^3,\ \ \frac{1}{\sqrt{2}},\ \ 1,\ \ 6/5$$

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