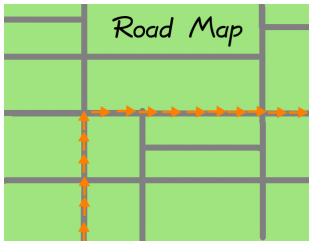
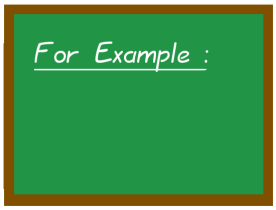
Module 1.7: Multiplying and Squaring Polynomials or Functions



In this module we're going to review the skill of multiplying two polynomials or two functions. This includes squaring a function (usually a polynomial) as a special case. This is relatively easy material but it will prove useful later in the book.

Since this topic is a bit algebraic, you might be curious as to what financial, scientific, or economic situations would call for using this. With that in mind, I've taken the unusual step of opening this module with a relatively complicated but realistic look at how to balance market forces in setting the best price for one night in an airport hotel. The heart of this example involves the multiplication of two functions.

After that example, we will be confident that this skill is actually used frequently in the book for realistic problems. Therefore, we will proceed to practice the skill for several pages, and conclude with another realistic example involving an industrial supply-chain management problem in the style of "lean manufacturing."



1-7-1

Let us imagine that you are an intern at an airport hotel. Competition surrounds your hotel, so you have to be careful about raising prices—if they become too high, then everyone will go to the surrounding hotels. Nonetheless, yours has a closer location than most of the others and does well. Your boss is curious if raising or lowering the price would produce increased revenue. Right now the rooms go for $ 89 per night. The boss says that "for every dollar we raise the nightly rate, we lose 7 customers." Currently, the hotel has 271 rooms full on average on a typical weeknight, out of 500 rooms. Let x be the number of dollars that the hotel raises the price. What will the hotel's revenue be, as a function of x?

To solve this problem, we first ask ourselves, if we raise the price by x dollars, what will the price be? Surely the price will be $89+x$. Likewise, they lose 7 customers for each dollar that they raise the price, so they will lose $7x$ customers. They have 271 on a typical night with the current price, and so they will have $(271-7x)$ customers with the raised price. Now, if we have $(271-7x)$ customers each paying $(89+x)$ dollars, the total revenue is

$$f(x) = (89+x)(271-7x)$$

The previous example might be a bit surprising, because we don't normally think of multiplying two functions as a model of revenue in a hotel. Let's look at the following chart:

Price	Customers	Revenue
89	271	$89 \times 271 = 24,119$
90	$271 - 7 = 264$	$90 \times 264 = 23,760$
91	$264 - 7 = 257$	$91 \times 257 = 23,387$
92	$257 - 7 = 250$	$92 \times 250 = 23,000$
93	$250 - 7 = 243$	$93 \times 243 = 22,599$
94	$243 - 7 = 236$	$94 \times 236 = 22,184$
95	$236 - 7 = 229$	$95 \times 229 = 21,755$
96	$229 - 7 = 222$	$96 \times 222 = 21,312$
97	$222 - 7 = 215$	$97 \times 215 = 20,855$
98	$215 - 7 = 208$	$98 \times 208 = 20,384$
99	$208 - 7 = 201$	$99 \times 201 = 19,899$

Well, it is really clear that raising the price isn't raising revenue. Let's try lowering the price.

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Continuing with the previous box, we're going to see what happens if we lower the price instead of raising it.

Price	Customers	Revenue
89	271	$89 \times 271 = 24,119$
88	$271 + 7 = 278$	$88 \times 278 = 24,464$
87	$278 + 7 = 285$	$87 \times 285 = 24,795$
86	$285 + 7 = 292$	$86 \times 292 = 25,112$
85	$292 + 7 = 299$	$85 \times 299 = 25,415$
84	$299 + 7 = 306$	$84 \times 306 = 25,704$
83	$306 + 7 = 313$	$83 \times 313 = 25,979$
82	$313 + 7 = 320$	$82 \times 320 = 26,240$
81	$320 + 7 = 327$	$81 \times 327 = 26,487$
80	$327 + 7 = 334$	$80 \times 334 = 26,720$
79	$334 + 7 = 341$	$79 \times 341 = 26,939$

Generating this table sure is tedious. It is clear that we're moving in the right direction, but this is a ton of work, and it would make more sense to try a more analytical approach.



The price was

$$p(x) = 89 + x$$

and the number of customers was

$$n(x) = 271 - 7x$$

and so the revenue will be

$$r(x) = p(x) \cdot n(x) = (89 + x)(271 - 7x)$$

which we can rewrite as

$$r(x) = (89 \cdot 271) - (89 \cdot 7)x + 271x - 7x^2$$

and our calculator helps us rewrite this as

$$r(x) = 24,119 - 352x - 7x^2$$

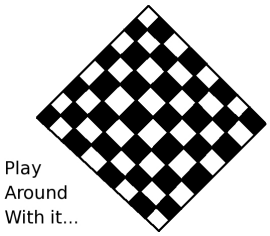


We can check our polynomial multiplication by plugging in values that happen to be from our table. Observe,

$$\begin{array}{rcccl} r(-6) & = & 24,119 - 352(-6) - 7(-6)^2 & = & 25,979 \quad \leftarrow \text{matches a price of \$ 83} \\ r(-2) & = & 24,119 - 352(-2) - 7(-2)^2 & = & 24,795 \quad \leftarrow \text{matches a price of \$ 87} \\ r(+2) & = & 24,119 - 352(+2) - 7(+2)^2 & = & 23,387 \quad \leftarrow \text{matches a price of \$ 91} \\ r(+6) & = & 24,119 - 352(+6) - 7(+6)^2 & = & 21,755 \quad \leftarrow \text{matches a price of \$ 95} \end{array}$$

Note that if we raise the price by \$ 6, then $x = 6$. Likewise the \$ 89 price becomes \$ 95. This is why we want $r(6)$ to equal the revenue for a price of \$ 95.

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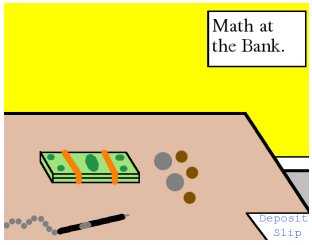


1-7-2

Later in this book, you'll learn how to find the optimum value of a parabola. Let's imagine you ask your friend Alice who took the course last semester. She tells you that the optimum price is \$ 64. The easiest way to verify this is to plug in \$ 63, \$ 64, and \$ 65, and then see what happens. These correspond to values of $x = -26$, $x = -27$, and $x = -28$.

- What value of x corresponds to a price of \$ 65? [Answer: $x = -24$.]
- What is $r(-24)$? [Answer: $r(-24) = 28,535$.]
- What value of x corresponds to a price of \$ 64? [Answer: $x = -25$.]
- What is $r(-25)$? [Answer: $r(-25) = 28,544$.]
- What value of x corresponds to a price of \$ 63? [Answer: $x = -26$.]
- What is $r(-26)$? [Answer: $r(-26) = 28,539$.]

As you can see, the revenue of 28,544 (per night!) is higher than any revenue claimed in this box or in any of our previous tables. It seems to be the winner. When we study this topic properly later in the book, you can have more confidence than guess-and-check can provide, because you will learn formulas that will reduce all this work, over the last several boxes, into some short and compact formulas.



The nightly revenue was before \$ 24,119 and would become \$ 28,544 if we follow Alice's plan. That sounds like a very small difference. Actually, it is \$ 8,803,435 versus \$ 10,418,560 for a 365-day year. For most business operations, \$ 1,615,125 is nothing to sneeze at.

Furthermore, this is only revenue. If the expenses are perhaps 9.5 million dollars per year, then this is the difference between a loss of \$ 696,565 versus a profit of \$ 918,560. That's the kind of difference that can determine whether a manager gets promoted or gets fired.

Hopefully, at this point, you are convinced that polynomial multiplication can come up in a business context. There's another fun example involving manufacturing later in this module, towards the end. Since we are convinced of this fact, we can focus on acquiring the skill, which will be regrettably algebraic and drill-like.



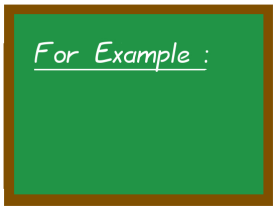
The way I multiply polynomials (when I'm working on some difficult calculus problem, for example) is by what is commonly called "the handshake method." This is an analogy to the shaking of hands after a sports game. After a hockey game, and the same is true for several other sports, it is traditional for each member of one team to shake hands with each member of the other team. It should be noted that

1. No one shakes hands with members of her/his own team.
2. Every possible pair of players (one from each team) will actually shake hands.
3. No one shakes hands with the same person more than once.

This is precisely what must be done when multiplying two polynomials. Every monomial in one of the polynomials must be multiplied by every monomial in the other polynomial, exactly once. The multiplication takes the role of shaking hands. Furthermore, monomials from the same polynomial do not get multiplied (1). Also, you cannot skip (2) or duplicate (3) any of the multiplications.

This is one of those techniques in math that is *far easier* to demonstrate than to explain. Accordingly, let's look at an example.

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1-7-3

Suppose we're given

$$f(x) = 3x^2 + 5x - 9$$

as well as

$$g(x) = 4x - 11$$

and we have to multiply them to get $h(x)$. That's like requesting the computation of

$$h(x) = f(x) \cdot g(x) = (3x^2 + 5x - 9)(4x - 11)$$

First, we'll have $4x$ shake hands with each term of $f(x)$ and then we'll have -11 shake hands with each term of $f(x)$. If we take that approach, we'd have

$$h(x) = \underbrace{12x^3 + 20x^2 - 36x}_{\text{from } 4x} + \underbrace{-33x^2 - 55x + 99}_{\text{from } -11}$$

and then we can combine the like terms to obtain

$$h(x) = 12x^3 + (20 - 33)x^2 + (-36 - 55)x + 99$$

and that simplifies to

$$h(x) = 12x^3 - 13x^2 - 91x + 99$$



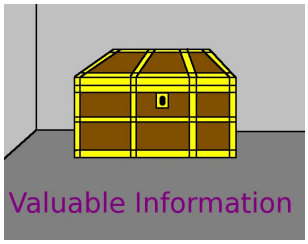
If it is the case that $h(x)$ is equal to $f(x)$ times $g(x)$, for all x in general, then it must be the case that $h(1)$ is equal to $f(1)$ times $g(1)$, in particular. Likewise, it should be the case that $h(0)$ is equal to $f(0)$ times $g(0)$ and similarly that $h(2)$ is equal to $f(2)$ times $g(2)$. After all, 0, 1, and 2, are just numbers on the number line. However, it is remarkably easy to plug in 0 and 1, and only slightly harder to plug in 2. Since this trio of substitutions is so easy to use, then it is a great way for us to check our work. Let's do that now, in the next box.



Here we will check our work from the previous example.

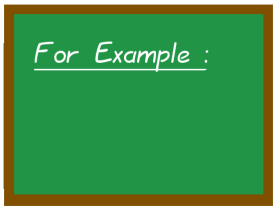
- Using $x = 0$ we get $f(0) = -9$, $g(0) = -11$, $h(0) = 99$, and that's good because $(-9)(-11) = 99$.
- Using $x = 1$ we get $f(1) = -1$, $g(1) = -7$, $h(1) = 7$, and that's good because $(-1)(-7) = 7$.
- Using $x = 2$ we get $f(2) = 13$, $g(2) = -3$, $h(2) = -39$, and that's good because $(13)(-3) = -39$.

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There are some words which we use in math that are really handy for us in the study of this topic.

- A polynomial of four terms, such as $12x^3 - 13x^2 - 91x + 99$ or $17x^5 + 256x^4 + 15x^3 - 153x^2$, is called a *quadrinomial*.
- A polynomial of three terms, such as $-13x^2 - 91x + 99$ or $17x^5 + 15x^3 - 153x^2$ is called a *trinomial*.
- A polynomial of two terms, such as $-13x^2 + 99$ or $17x^5 + 15x^3$ is called a *binomial*.
- A polynomial of one term, such as $-13x^2$ or $17x^5$ is called a *monomial*.
- The individual terms inside a long polynomial, namely the $12x^3$ and the $-13x^2$, as well as the $-91x$ and the 99, belonging to $12x^3 - 13x^2 - 91x + 99$, are also called monomials.



1-7-4

Suppose we're given

$$a(x) = 17x + 1$$

as well as

$$b(x) = x^4 + 15x^3 - 9x$$

and we have to multiply them. That's like requesting the computation of

$$c(x) = a(x) \cdot b(x) = (17x + 1)(x^4 + 15x^3 - 9x)$$

First, we'll have $17x$ shake hands with each term of $b(x)$ and then we'll have $+1$ shake hands with each term of $b(x)$. If we take that approach, we'd have

$$c(x) = \underbrace{17x^5 + 255x^4 - 153x^2}_{\text{from } 17x} + \underbrace{x^4 + 15x^3 - 9x}_{\text{from } +1}$$

noting that $255 = 17 \times 15$ and $153 = 17 \times 9$. Next, we can combine the like terms to obtain

$$c(x) = 17x^5 + (255 + 1)x^4 + 15x^3 - 153x^2 - 9x$$

and that simplifies to

$$c(x) = 17x^5 + 256x^4 + 15x^3 - 153x^2 - 9x$$

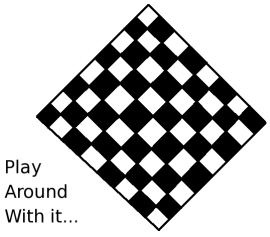


Using our strategy of checking our work with 0, 1, and 2, we see the following.

- Using $x = 0$ we get $a(0) = 0$, $b(0) = 0$, $c(0) = 0$, and that's good because $(0)(0) = 0$.
- Using $x = 1$ we get $a(1) = 18$, $b(1) = 7$, $c(1) = 126$, and that's good because $(18)(7) = 126$.
- Using $x = 2$ we get $a(2) = 35$, $b(2) = 118$, $c(2) = 4130$, and that's good because $(35)(118) = 4130$.

Notice, you can save yourself a little effort if you organize things according to the polynomial with fewer terms. That's why we expanded according to the binomial, and not the trinomial.

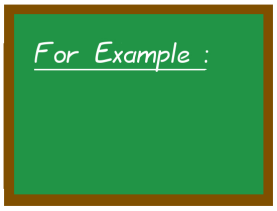
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1-7-5

Now give these a try, to see if you've gotten the technique.

- What is $(x^2 + x - 2)(3x^2 - x + 1)$? [Answer: $3x^4 + 2x^3 - 6x^2 + 3x - 2$.]
- What is $(x^2 + 5x - 12)(x^2 + 3x + 4)$? [Answer: $x^4 + 8x^3 + 7x^2 - 16x - 48$.]
- Note: if you got the first two correct, you don't have to do the third one.
- What is $(x^2 - 9)(x^2 - x + 2)$? [Answer: $x^4 - x^3 - 7x^2 + 9x - 18$.]



1-7-6

Let's consider the special case of a binomial times a binomial. Using our handshake method we could look at two examples.

$$(2x - 3)(4x + 5) = \underbrace{8x^2 + 10x}_{\text{from } 2x} \underbrace{-12x - 15}_{\text{from } -3}$$

and then

$$(8x + 1)(3x + 4) = \underbrace{24x^2 + 32x}_{\text{from } 8x} \underbrace{+3x + 4}_{\text{from } +1}$$

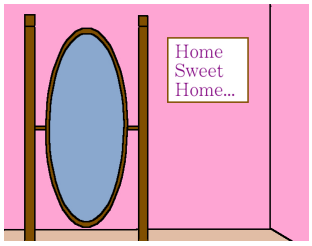
It was the case here, and will always be the case, that the first of the four terms was the product of the "first" terms of each binomial. Likewise, the second of the four terms was the product of the "outer" terms of each binomial. Similarly, the third of the four terms was the product of the "inner" terms of each binomial. Finally, the fourth of the four terms was the product of the "last" terms of each binomial. This pattern of first-outer-inner-last can be abbreviated F.O.I.L. or "foil." Many students use the word "foil" as a memory hook.

Therefore, we can relabel as follows:

$$(2x - 3)(4x + 5) = \underbrace{8x^2}_{\text{first}} + \underbrace{10x}_{\text{outer}} - \underbrace{12x}_{\text{inner}} - \underbrace{15}_{\text{last}}$$

as well as

$$(8x + 1)(3x + 4) = \underbrace{24x^2}_{\text{first}} + \underbrace{32x}_{\text{outer}} + \underbrace{3x}_{\text{inner}} + \underbrace{4}_{\text{last}}$$

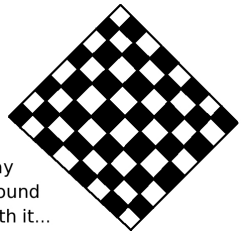


A Pause for Reflection...

Personally, it has always struck me as a bit odd that students will choose to have a special word or memory hook "foil" for the particular case of a binomial times a binomial. The foil concept does not work if either of the polynomials or functions is a trinomial or longer, nor if one of them is a monomial. Most of the times, therefore, you cannot apply foil. Does it make sense to have a separate technique and a separate memory hook that only works in one special case? On the other hand, the binomial-times-a-binomial case is relatively common.

I think it is better to focus on either the handshake method (which you've already seen) or the factor box method (which I'm about to show you). Of course, you can use whatever methods you feel comfortable with. If you like the foil method, then you should feel free to use it.

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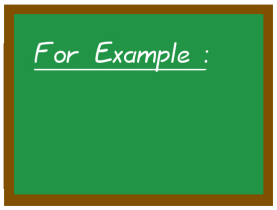


Play Around With it...

1-7-7

Here are four cases of a binomial times a binomial. You can feel free to use either the handshake method or the foil method, as you like.

- If $f(x) = 3x + 5$ and $b(x) = 4x - 1$ then what is $a(x) \cdot b(x)$? [Answer: $12x^2 + 17x - 5$.]
- If $f(x) = 2x - 6$ and $b(x) = 9x + 1$ then what is $a(x) \cdot b(x)$? [Answer: $18x^2 - 52x - 6$.]
- If $f(x) = x^2 + 8$ and $b(x) = 3 - x$ then what is $a(x) \cdot b(x)$? [Answer: $-x^3 + 3x^2 - 8x + 24$.]



1-7-8

Here is a wacky example: if $a(x) = 5x - 3$ and $b(x) = 4 + \sqrt{x}$ then compute $c(x) = a(x) \cdot b(x)$.

At first, this looks like the foil method, where

$$(5x - 3)(4 + \sqrt{x}) = \underbrace{20x}_{\text{first}} + \underbrace{5x\sqrt{x}}_{\text{outer}} - \underbrace{12}_{\text{inner}} - \underbrace{3\sqrt{x}}_{\text{last}}$$

and so we conclude

$$c(x) = 12 + 3\sqrt{x} + 20x + 5x\sqrt{x}$$

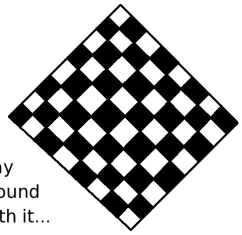
and that's true.

The $5x\sqrt{x}$ certainly looks weird, but this sort of arrangement does occur. The square root function can come up in economics for storage/warehousing costs, and certain famous problems, like laying pipe, can involve a square root because of the Pythagorean Theorem. We'll cover all this later in the book, so don't worry about it now. The only thing that I want to bring to your attention is that $c(x)$ is *not a polynomial*.



If this were a pure math class, I'd spend time and illustrate this point by identifying properties of $c(x)$ that no polynomial would ever have. However, that's not of interest to us in this course. Still, $c(x)$ is a perfectly legitimate function and it is because of this category of example that I have "or functions" in the title of this module.

By the way, you should also know that $x\sqrt{x}$ is sometimes abbreviated as $x^{3/2}$ or $x^{1.5}$. However, I haven't taught you that yet. We'll learn that on Page 352. The only reason I mention it now is if you are working with a tutor (or a computer algebra system), and you see those symbols then I want you to know what those symbols mean.



Play Around With it...

1-7-9

- If $f(x) = 4x + 9$ and $b(x) = 2 + \sqrt{x}$ then what is $a(x) \cdot b(x)$? [Answer: $4x\sqrt{x} + 8x + 9\sqrt{x} + 18$.]
- For the next one, note that $\sqrt{x}\sqrt{x} = x$.
- If $f(x) = 5 + \sqrt{x}$ and $b(x) = 6 + \sqrt{x}$ then what is $a(x) \cdot b(x)$? [Answer: $x + 11\sqrt{x} + 30$.]

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For Example :

1-7-10

We're going to multiply now the following two polynomials, but with the factor box method:

Find $c(x) = a(x) \cdot b(x)$ where $a(x) = 4x^2 - 7x + 11$ and $b(x) = -x^2 + 3x - 9$

What we're going to do is to construct a grid of boxes called factor boxes. Here is how it will look.

	$-x^2$	$+3x$	-9
$4x^2$			
$-7x$			
$+11$			

The key components are that one polynomial is filling in the entries of the left-hand column, and the other polynomial is the top row. Also, it is very useful to have the terms go from highest degree to lowest degree.

Okay, now it is time to fill-in the entries, which we will do in the next box.

For each empty spot, we're going to take the entry at the top of the row, and the entry at the left of the column, multiply those, and write the product in the empty spot.

- $(-x^2)(+4x^2) = -4x^4$.
- $(-x^2)(-7x) = +7x^3$.
- $(-x^2)(+11) = -11x^2$.
- $(+3x)(+4x^2) = +12x^3$.
- $(+3x)(-7x) = -21x^2$.
- $(+3x)(+11) = +33x$.
- $(-9)(+4x^2) = -36x^2$.
- $(-9)(-7x) = +63x$.
- $(-9)(+11) = -99$.

This gives the final answer of

	$-x^2$	$+3x$	-9
$4x^2$	$-4x^4$	$+7x^3$	$-11x^2$
$-7x$	$+12x^3$	$-21x^2$	$+33x$
$+11$	$-36x^2$	$+63x$	-99

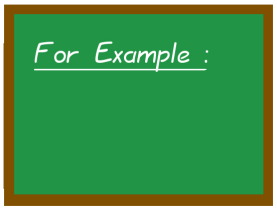
Now the final answer could be written as the sum of all the inside entries. In this case

$$c(x) = -4x^4 + 7x^3 + 12x^3 - 36x^2 - 21x^2 - 11x^2 + 33x + 63x - 99$$

is one way of writing the final answer, but it isn't very useful. We'd want to combine like terms and get

$$c(x) = -4x^4 + (7 + 12)x^3 + (-36 - 21 - 11)x^2 + (33 + 63)x - 99 = -4x^4 + 19x^3 - 68x^2 + 96x - 99$$

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1-7-11

Now let's say that you're asked to multiply the following functions, with the factor-box method.

Find $h(x) = f(x) \cdot g(x)$ where $f(x) = 3x^2 - 9x + 45$ and $g(x) = 3x^2 - 1$

We would setup the factor boxes like this:

	$3x^2$	$+0x$	-1
$3x^2$			
$-9x$			
$+45$			

Did you notice how we put a $0x$ as the middle column? This is not strictly necessary but I think it is quite helpful. It creates a "striped pattern" that I will tell you about in the box after the next.

Continuing with the previous example, now we can fill-in the factor boxes. We obtain

	$+3x^2$	$+0x$	-1
$+3x^2$	$+9x^4$	0	$-3x^2$
$-9x$	$-27x^3$	0	$+9x$
$+45$	$+135x^2$	0	-45

Our final answer is

$$h(x) = 9x^4 + (0 - 27)x^3 + (-3 + 0 + 135)x^2 + (0 + 9)x - 45 = 9x^4 - 27x^3 + 132x^2 + 9x - 45$$



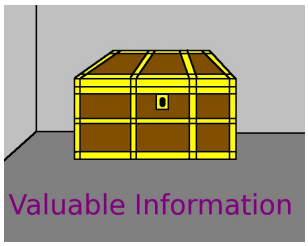
There is a pattern present in both the previous examples. Did you notice how, in the long three-box diagonal going from the lower-left to the upper-right, we had all quadratics? Then in the two-box diagonal above it, we have only cubics. Also, in the two-box diagonal below it, we had only linear monomials. This will always happen, so long as you represent any missing coefficients with 0. Of course, most polynomials are not missing coefficients.

Also, the sign pattern for the top row of the first example is $- + -$. That means all other rows should be either $- + -$ or the perfect opposite, $+ - +$. You will see that is the case. Similarly, the sign pattern for the top row of the second example is $+ + -$. That means all other rows should be either $+ + -$ or $- - +$. This is a great way to check your work.

Following the same pattern, the sign pattern of the leftmost column in the first example was $+ - +$, and so all other columns must be $+ - +$ or $- + -$. Lastly, the sign pattern of the leftmost column in the second example is also $+ - +$ and so all other columns must be $+ - +$ or $- + -$.

These sign patterns give rise to a great way of checking your work.

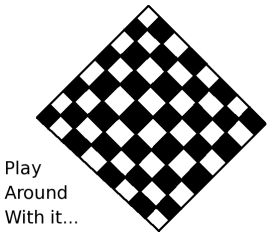
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When multiplying polynomials via the factor box method...

- Pick any row and look at its sign pattern (the pattern of positives and negatives). Every row has to have exactly the same pattern, or exactly the opposite.
- Pick any column and look at its sign pattern. Every column has to have exactly the same pattern, or exactly the opposite.

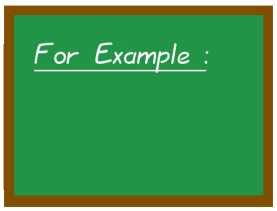
This check is good, because it detects errors where a sign change has been made accidentally. It is easy to drop a plus or a minus, and this will catch that. However, it does not catch numerical errors in the multiplication. For that, you should plug in 0 and 1.



1-7-12

For these, please use the factorbox method.

- What is $(x^2+x-2)(8x^2+x+1)$? [Answer: $8x^4+9x^3-14x^2-x-2$.]
- What is $(x^2+6x-18)(x^2+8x+4)$? [Answer: $x^4+14x^3+34x^2-120x-72$.]
- Note: if you got the first two correct, you don't have to do the third one.
- What is $(x^2+5)(x^2-x-7)$? [Answer: $x^4-x^3-2x^2-5x-35$.]



1-7-13

Let's now see what would happen if you were asked to square a polynomial, such as

$$r(x) = 4x^2+3x+2$$

using the handshake method.

It turns out that this is no different from a polynomial multiplication. It is like multiplying two polynomials with the handshake method, except that the two polynomials happen to be twins—they are both $4x^2+3x+2$.

$$\begin{aligned}(4x^2+3x+2)(4x^2+3x+2) &= \underbrace{+16x^4+12x^3+8x^2}_{\text{from } 4x^2}\underbrace{+12x^3+9x^2+6x}_{\text{from } +3x}\underbrace{+8x^2+6x+4}_{\text{from } +2}\\ &= 16x^4+(12+12)x^3+(8+9+8)x^2+(6+6)x+4\\ &= 16x^4+24x^3+25x^2+12x+4\end{aligned}$$

So it is no different from any other handshake-method polynomial multiplication.

One way to write "the square of $r(x)$" is to write $(r(x))^2$ but we can abbreviate this as $r^2(x)$. This does look kind of strange, to have the 2 stuck there between the r and the parentheses, but it is a common abbreviation.

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We can check our work from the previous example with 0, 1, and 2, as before.

- Because we get 4 when we plug $x = 0$ into $r(x)$, we expect $r^2(0) = 4^2 = 16$. This is true, because

$$16(0)^4 + 24(0)^3 + 25(0)^2 + 12(0) + 4 = 0 + 0 + 0 + 0 + 4 = 4$$

- Because we get 9 when we plug $x = 1$ into $r(x)$, we expect $r^2(1) = 9^2 = 81$. This is true, because

$$16(1)^4 + 24(1)^3 + 25(1)^2 + 12(1) + 4 = 16 + 24 + 25 + 12 + 4 = 81$$

- Because we get 24 when we plug $x = 2$ into $r(x)$, we expect $r^2(1) = 24^2 = 576$. This is true, because

$$16(2)^4 + 24(2)^3 + 25(2)^2 + 12(2) + 4 = 256 + 192 + 100 + 24 + 4 = 576$$



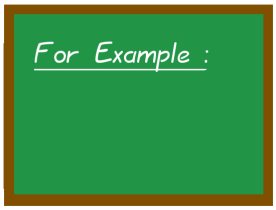
A misunderstanding that is not uncommon is that students will confuse $r^2(x)$ with $r(x^2)$. Let's consider when $x = 2$. We saw in the previous box that $r^2(2) = 576$.

What then would $r(2^2)$ be? Well, because $2^2 = 4$, that would be $r(4)$. We would then compute

$$r(4) = 4(4)^2 + 3(4) + 2 = 64 + 12 + 2 = 78$$

Because $576 \neq 78$ then we know for sure that $r^2(2) \neq r(2^2)$. This also implies that it is not the case that $r(x^2) = r^2(x)$ for all x, since we've found an exception, namely $x = 2$.

The moral of the story is to watch—very carefully—where you put your parentheses.



1-7-14

Let's say that you're asked to square the following polynomial.

$$f(x) = x^2 + 2x + 3$$

This is like being asked to multiply two polynomials—just like we've been doing a whole bunch of times in the last several pages. It is just that coincidentally, both polynomials happen to be $x^2 + 2x + 3$. With that in mind, we construct the factor table like so:

	$+x^2$	$+2x$	$+3$
$+x^2$			
$+2x$			
$+3$			

Give it quick try yourself. Then in the next box, we will do it together.

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The factor boxes from the previous example naturally turn into

	$+x^2$	$+2x$	$+3$
$+x^2$	$+x^4$	$+2x^3$	$+3x^2$
$+2x$	$+2x^3$	$+4x^2$	$+6x$
$+3$	$+3x^2$	$+6x$	$+9$

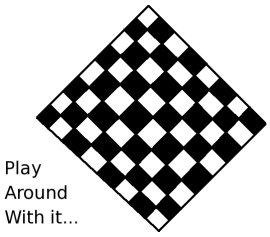
which means that the final answer is

$$f^2(x) = x^4 + (2+2)x^3 + (3+4+3)x^2 + (6+6)x + 9 = x^4 + 4x^3 + 10x^2 + 12x + 9$$

By the way, I will put a plus sign on every positive term in a factor-box-method problem. Why do I do this? Because almost all the errors that I see when grading are issues involving a positive becoming negative or a negative becoming positive. Since this is the most sensitive part of the entire process, making an extra plus sign now and again to keep everything clear in the mind's eye is a worthy investment of a fraction of a second, especially if it can result in a higher quiz or exam score.



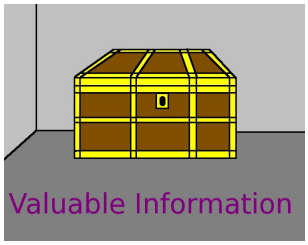
It is not a terribly important point, but whenever you square a polynomial using the factor-box method, you will get factor boxes that are symmetric in a particular way. The first column equals the first row, the second column equals the second row, the third column equals the third row, ..., and the last column equals the last row.



1-7-15

For these, you may use either the factor-box method or the handshake method, as you prefer.

- If $a(x) = x^2 + x + 5$ then what is $a^2(x)$? [Answer: $x^4 + 2x^3 + 11x^2 + 10x + 25$.]
- If $b(x) = 2x^2 + 3x + 4$ then what is $b^2(x)$? [Answer: $4x^4 + 12x^3 + 25x^2 + 24x + 16$.]
- If you have gotten the first two correct, you need not do the third one.
- If $c(x) = x^2 - x + 6$ then what is $c^2(x)$? [Answer: $x^4 - 2x^3 + 13x^2 - 12x + 36$.]

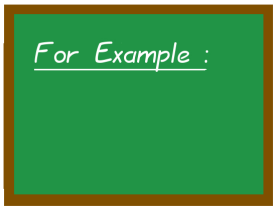


The following identity

$$(a+b)^2 = a^2 + 2ab + b^2$$

can be used to rapidly square a binomial. Note that a and b can be numbers, variables, complicated expressions, or anything in between.

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1-7-16

Using the formula from the previous box, we can square binomials extremely rapidly. Here are three examples.

$$(4x+7)^2 = \underbrace{(4x)^2}_{a^2} + \underbrace{2(4x)(7)}_{2ab} + \underbrace{7^2}_{b^2} = 16x^2 + 56x + 49$$

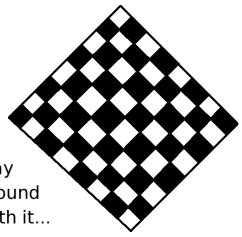
as well as

$$(3x-5)^2 = \underbrace{(3x)^2}_{a^2} + \underbrace{2(3x)(-5)}_{2ab} + \underbrace{(-5)^2}_{b^2} = 9x^2 - 30x + 25$$

or perhaps

$$(-x-9)^2 = \underbrace{(-x)^2}_{a^2} + \underbrace{2(-x)(-9)}_{2ab} + \underbrace{(-9)^2}_{b^2} = x^2 + 18x + 81$$

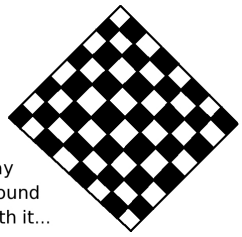
As you can see, it is a straight forward process—the trick is getting the positives and negatives to come out correctly.



1-7-17

You can use any method you want on these, but I recommend you try the shortcut!

- If $f(x) = 3x - 1$ then what is $f^2(x)$? [Answer: $9x^2 - 6x + 1$.]
- If $g(x) = 5 - x$ then what is $g^2(x)$? [Answer: $25 - 10x + x^2$.]
- If $h(x) = -2 - 6x$ then what is $h^2(x)$? [Answer: $4 + 24x + 36x^2$.]



1-7-18

You can use any method you want on these, but I recommend you try the shortcut!

- If $r(x) = x + 4\sqrt{x}$ then what is $r^2(x)$? [Answer: $x^2 + 8x\sqrt{x} + 16x$.]
- If $s(x) = 9 + 3\sqrt{x}$ then what is $s^2(x)$? [Answer: $81 + 54\sqrt{x} + 9x$.]

Thank you for being patient through that long series of mechanical algebraic computations. Now we can return to the fun world of problem solving, and we're going to look at a manufacturing problem.

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For Example :

1-7-19

A new manufacturing process for a rare but profitable transistor has been recently patented by your company. A production line has been set up, and the equipment has been purchased. The main monthly expense at this point is a particularly expensive type of solvent. The engineers running the production line say that they need 486 liters of solvent, and expect that the requirement will increase by 11 liters per month. The price of the solvent is 51 dollars per liter and is expected to rise by 2 dollars per month. (Of course, these simplistic models can only be used in the near future, not the far future, much like predicting the weather. The weather forecast for tomorrow is often accurate, but five days out is rarely an accurate forecast.) What function $f(t)$ represents the total expenditure in dollars for the tth month, where $t = 0$ signifies this month?

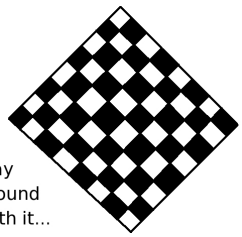
As a warmup, for this month, 486 liters are needed and each liter costs 51 dollars, so the need is

$$486 \times 51 = 24,786 \text{ dollars per month}$$

In the general case, since they are using up 486 liters of solvent, and the requirement will increase by 11 liters per month, we would write that $(486 + 11t)$ liters should be used in the tth month. Likewise, the price will be $(51 + 2t)$ dollars per liter. Then we obtain:

$$f(t) = (486 + 11t)(51 + 2t)$$

is the cost for the solvents in the tth month.



1-7-20

Before we continue, take a moment to do the following

- By any means that you like, multiply out $f(t)$ from the previous box. [Answer: $f(t) = 24,786 + 1533t + 22t^2$.]
- Without looking at the polynomials, what should the cost per liter be in the third month? [Answer: \$ 57 per liter.]
- In the third month, how many liters are required ? [Answer: 519 liters.]
- From the previous bullet, what do you expect the cost in dollars to be for the third month? [Answer: $519 \times 57 = \$\ 29,583$.]
- What is $f(3)$? [Answer: $f(3) = 24,786 + 4599 + 198 = 29,583$.]

For Example :

1-7-21

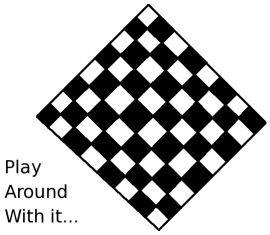
It turns out that the company from the previous example is owned by a British conglomerate. The conglomerate has its accounting offices in the greater London area, and so they track every cost in pounds sterling. Currently, a dollar is 0.644 pounds sterling, but this is expected to rise at the rate of 0.031 pounds sterling per month. How can we write a new cost function in pounds sterling, $g(t)$, that reflects the changing currency price.

As a warmup, if the exchange rate were locked at 0.644 pounds per dollar, we could just multiply $f(t)$ by 0.644 and be done. However, the exchange rate is not locked at 0.644 pounds per dollar, but will be $(0.644 + 0.031t)$ because the currency fluctuation. We then have

$$g(t) = (0.644 + 0.031t)f(t) = (0.644 + 0.031t)(24,786 + 1533t + 22t^2)$$

which will turn out to be a cubic function.

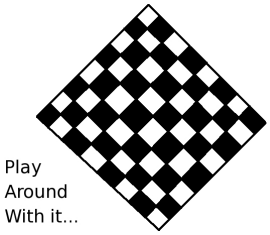
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1-7-22

Before we continue, take a moment to do the following. By any means that you like, multiply out $g(t)$ from the previous box.

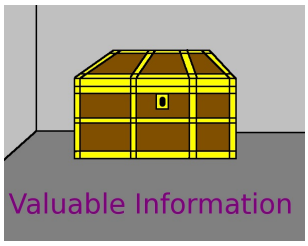
$$\left[\text{Answer: } 15,962.18 + (1755.61\cdots)t + 61.691t^2 + 0.682t^3.\right]$$



1-7-23

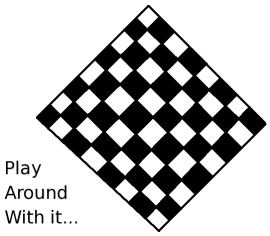
- Without looking at that polynomial, what do you expect the exchange rate to be 3 months from now? [Answer: \$ 0.737 per pound sterling.]
- Knowing (from the previous checkerboard) that \$ 29,583 will be spent on solvent in Month 3, how many pounds is that? [Answer: 21,802.67 pounds.]
- What is $g(3)$?
  [Answer: $g(3) = 15,962.1\cdots + 5266.85\cdots + 555.219\cdots + 18.414 = 21,802.6\cdots$.]

Note: since we don't have enough significant figures to write the pennies, we will not. We can write either \$ 21,802 or \$ 21,803.



You might find it amusing to note that actually I, when typing the above problem, myself made a major error. I caught it when checking my work by computing $g(3)$. I tell you this so that you really understand that professionals check their work—it is the only way to have confidence in your computations. As it turns out, I had typed three fives, instead of two fives, when typing 1755.61, and that error propagated through the problem.

Always, always check your work!



1-7-24

There's one last interesting point to share. In the end, you could write $g(t)$ as follows

$$g(t) = (0.644 + 0.031t)(486 + 11t)(51 + 2t)$$

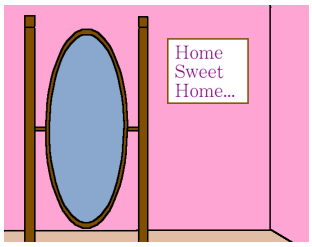
and then if you needed to multiply it out, perhaps you'd multiply the first two binomials, and then multiply that answer by the last binomial. This is not the order in which we multiplied the polynomials in the last few boxes.

- Multiply out $(0.644 + 0.031t)(486 + 11t)$. [Answer: $312.984 + 22.15t + 0.341t^2$.]
- Now multiply the answer from the previous bullet by $(51 + 2t)$.

$$\left[\text{Answer: } 15,962.18 + (1755.61\cdots)t + 61.691t^2 + 0.682t^3.\right]$$

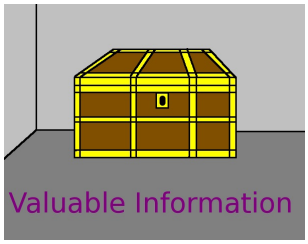
Therefore, we learn that we can multiply the binomials in any order. This makes sense, as we can multiply integers and fractions in any order also.

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A Pause for Reflection...

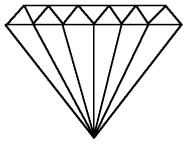
I think it is really neat that a cubic polynomial has come up in a cost-forecasting problem, and in a natural way. When we learn about regressions, on Page 434, then we will see that often in business a quadratic polynomial can be constructed to represent data: such as the amount of solvent needed, the price of solvent, or the exchange rate. Multiplying three quadratic polynomials together, as we would need to do to predict the amount of money (in pounds sterling) required to purchase the solvent required, would yield a degree-six polynomial, probably with all seven terms having an ugly decimal number. Sometimes models are compact and beautiful, and sometimes they are rather complicated. That's just how things are—not everything in the real world is neat, clean, and easy.



The subject of *lean manufacturing* has many aspects to it, but some of them are to only use what you need (avoiding waste); to only purchase as much supplies as you need (avoiding large stockpiles); to purchase them exactly when you need them (avoiding warehousing); to only budget money for what you need to purchase (which avoids having capital sit idle.)

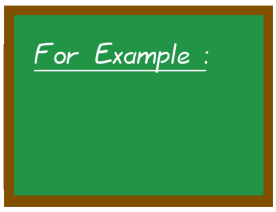
Lean manufacturing is a new and rapidly growing subject.

Hard but Valuable!



Now we're going to look at a few puzzle-like problems that will be a fun game. The following problems will look really weird, and are not representative of any situation that will occur later in applied mathematics or even pure math. These problems are meant to challenge you and to see if you've really grasped the internal mechanics of multiplying polynomials and functions. It will be a brief, and hopefully enjoyable, excursion.

Some instructors will skip this material but I hope you will read it and try it anyway.



1-7-25

The following factor box is peculiar in that many of the entries have been removed and replaced with question marks. Our goal is to try to recover the original polynomials, and then compute the final answer.

	$+2x^2$	?!?	?!?
?!?	$-6x^4$	?!?	$-42x^2$
$+8x$	?!?	$-88x^2$	?!?
-9	?!?	?!?	?!?

Try, yourself, to find some the ?!? entries, and then we'll solve it together in the next box.

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We can now make the following three observations:

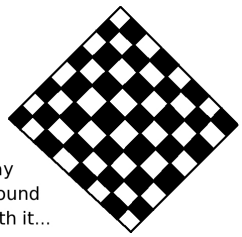
- The entry $-88x^2$ is a product of $+8x$ with an unknown entry, but that entry could only be $-11x$ so that $(+8x)(-11x) = -88x^2$.
- The entry $-42x^2$ is a product of two unknowns, so we cannot use it yet.
- The entry $-6x^4$ is a product of $+2x^2$ with an unknown entry, but that entry could only be $-3x^2$ so that $(+2x^2)(-3x^2) = -6x^4$.
- Ah! Now we know that $-42x^2$ is a product of $-3x^2$ and something unknown. Of course, that unknown could only be $+14$ so that $(-3x^2)(+14) = -42x^2$.

We will continue in the next box.

Continuing with the previous box, at this point, the factor boxes look like this:

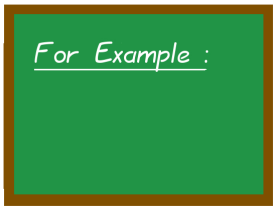
	$+2x^2$	$-11x$	$+14$
$-3x^2$	$-6x^4$	?!?	$-42x^2$
$+8x$	?!?	$-88x^2$	?!?
-9	?!?	?!?	?!?

As you can see, we now know both polynomials involved, because the entire top row and left column are filled in. Those polynomials are $2x^2 - 11x + 14$ and $-3x^2 + 8x - 9$. Therefore, we can just fill in the remaining boxes in the usual way—as if we knew the original polynomials only, but nothing else.



Finish filling in the factor boxes of the previous example. Then compute the final product of the given polynomials. The answer can be found on Page 219. Then we will do another example together before challenging you with a few examples from scratch.

1-7-26



Once again, we are presented with a factor box where many of the entries have been removed and replaced with question marks. As before, our goal is to try to recover the original question, and then compute the final answer.

	?!?	$+11x$	$+3$
$13x^2$	$+65x^4$	?!?	?!?
?!?	?!?	$-77x^2$	?!?
?!?	$100x^2$	?!?	?!?

1-7-27

Just like before, we'll solve it together in the next box, but feel free to give it a try yourself first, if you like.

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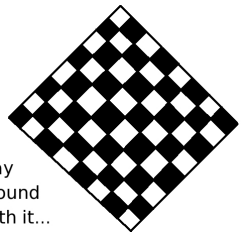
We can now make the following three observations:

- The entry $-77x^2$ is a product of $+11x$ with an unknown entry, but that entry could only be $-7x$ so that $(-7x)(+11x) = -77x^2$.
- The entry $100x^2$ is a product of two unknowns, so we cannot use it yet.
- The entry $+65x^4$ is a product of $+13x^2$ with an unknown entry, but that entry could only be $+5x^2$ so that $(+13x^2)(+5x^2) = +65x^4$.
- Now we know that $+100x^2$ is a product of $+5x^2$ and something unknown. Of course, that unknown could only be 20 so that $(+5x^2)(20) = 100x^2$.

At this point, the factor boxes look like this:

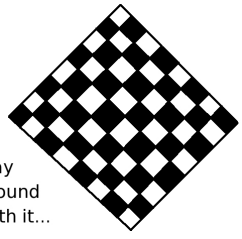
	$+5x^2$	$+11x$	$+3$
$+13x^2$	$+65x^4$	?!?	?!?
$-7x$	?!?	$-77x^2$	?!?
$+20$	$+100x^2$	?!?	?!?

As you can see, we now know both polynomials involved, because the entire top row and left column are filled in. There are $+13x^2 - 7x + 20$ and $+5x^2 + 11x + 3$, and therefore, we can fill them in as we did in the previous example.



1-7-28

Finish filling in the factor boxes of the previous example. Then compute the final product of the given polynomials. The answer can be found on Page 219. Then we will do another example together before challenging you with a few examples from scratch.



1-7-29

Now you get to try the entire process by yourself! The answer is to be found on Page 219.

1

	$+15x^2$	?!?	$+4$
$+2x^2$	?!?	$+18x^3$	?!?
$-x$	?!?	?!?	?!?
?!?	$+45x^2$	?!?	?!?

2

	$+4x^2$	?!?	?!?
$+2x^2$	?!?	?!?	$-2x^2$
?!?	$12x^3$	?!?	?!?
-7	?!?	$42x$	?!?

3

	?!?	$+9x$	$+11$
$+2x^2$	$-12x^4$	?!?	?!?
?!?	?!?	?!?	$11x$
?!?	?!?	$-45x$	?!?

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Here is the solution to the checkerboard on Page 217.

	$+2x^2$	$-11x$	$+14$
$-3x^2$	$-6x^4$	$+33x^3$	$-42x^2$
$+8x$	$+16x^3$	$-88x^2$	$+112x$
-9	$-18x^2$	$+99x$	-126

The final answer is, therefore,

$$-6x^4+(33+16)x^3+(-42-88-18)x^2+(112+99)x-126 = -6x^4+49x^3-148x^2+211x-126$$



Here is the solution to the checkerboard on Page 218.

	$+5x^2$	$+11x$	$+3$
$+13x^2$	$+65x^4$	$+143x^3$	$+39x^2$
$-7x$	$-35x^3$	$-77x^2$	$-21x$
$+20$	$+100x^2$	$+220x$	$+60$

The final answer is, therefore,

$$65x^4+(143-35)x^3+(100-77+39)x^2+(220-21)x+60 = 65x^4+108x^3+62x^2+199x+60$$



Here is the solution to #1 of the big checkerboard of three factor boxes, from Page 218.

1

	$+15x^2$	$+9x$	$+4$
$+2x^2$	$+30x^4$	$+18x^3$	$+8x^2$
$-x$	$-15x^3$	$-9x^2$	$-4x$
$+3$	$+45x^2$	$+27x$	$+12$

$30x^4 + 3x^3 + 44x^2 + 23x + 12$



Here is the solution to #2 of the big checkerboard of three factor boxes, from Page 218.

2

	$+4x^2$	$-6x$	-1
$+2x^2$	$+8x^4$	$-12x^3$	$-2x^2$
$+3x$	$+12x^3$	$-18x^2$	$-3x$
-7	$-28x^2$	$+42x$	$+7$

$8x^4 - 48x^2 + 39x + 7$



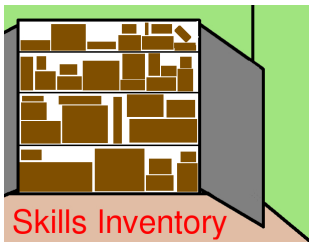
Here is the solution to #3 of the big checkerboard of three factor boxes, from Page 218.

3

	$-6x^2$	$+9x$	$+11$
$+2x^2$	$-12x^4$	$18x^3$	$22x^2$
$+x$	$-6x$	$9x$	$11x$
-5	$30x^2$	$-45x$	-55

$-12x^4 + 12x^3 + 61x^2 - 34x - 55$

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We have learned the following skills in this module:

- How to multiply polynomials with the handshake method.
- How to check your work when multiplying polynomials by plugging in 0, 1, or 2.
- How to multiply polynomials with the factor box method.
- How to multiply binomials with the "foil" method.
- We saw that these methods work with functions that are not polynomials, such as functions containing $\sqrt{x}$.
- How to apply these all methods when squaring polynomials (or other functions).
- How to square binomials with $(a+b)^2 = a^2 + 2ab + b^2$.
- We saw two examples of quadratic functions coming out of multiplying two linear models; we also so a cubic function come out that way.
- Optionally, how to recover all the entries in a completed factor box when most of those entries have been replaced by question marks.
- As well as the vocabulary terms: lean manufacturing, as well as quadrinomial, trinomial, binomial, and monomial.

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