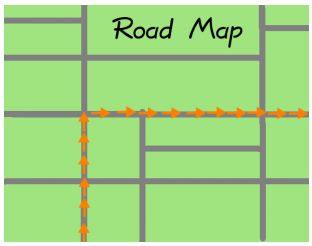
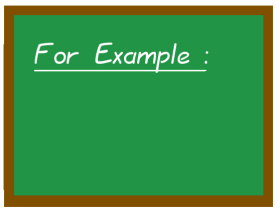
Module 3.10: The Number e and Continuously Compounded Interest



In this module, we're going to learn about *continuous compounding*, which is a more advanced form of compound interest. It will also be our first encounter with the amazing number e. We will discuss the properties of e in detail, and see how this number ties together all the exponential models we have been discussing during this chapter.



Suppose that I am saving up for 20 years, with some sort of savings bond, and that the interest rate is locked in at 5% per year, for the whole 20 years. I am depositing 1 million dollars, and the compounding period is going to be one of the choices below.

3-10-1

For each of the following compounding periods, and $P = 1,000,000$, what you get is the following:

Period of Compounding	m Periods per Year	$n = mt$ Number of Periods	$i = r/m$ Interest per Period	$A = P(1+i)^n$ Amount
Annually	1	20	$0.05/1 = 0.05$	2,653,297.71
Semiannually	2	40	$0.05/2 = 0.025$	2,685,063.84
Quarterly	4	80	$0.05/4 = 0.0125$	2,701,484.94
Bimonthly	6	120	$0.05/6 = 0.08333\cdots$	2,707,041.49
Monthly	12	240	$0.05/12 = 0.041666\cdots$	2,712,640.29
Biweekly	26	520	$0.05/26 = 0.00192307\cdots$	2,715,672.70
Weekly	52	1040	$0.05/52 = 0.000961538\cdots$	2,716,976.11
Daily	360	7200	$0.05/360 = 0.00013888\cdots$	2,718,093.08
Hourly	360×24	172,800	$0.05/(360 \times 24) = 5.78703\cdots \times 10^{-6}$	2,718,273.96
Minutely	$(360)(24)(60)$	10,368,000	$0.05/(360 \times 24 \times 60) = 9.64506\cdots \times 10^{-8}$	2,718,281.70
Secondly	$(360)(24)(60)(60)$	622,080,000	$0.05/(360 \times 24 \times 60^2) = 1.60751\cdots \times 10^{-9}$	2,718,281.92

Now surely, the last three compounding periods are just fictional. No one, except possibly a mafia loan-shark, would compound interest hourly. They are printed here to prove a point: observe that as you go down the table, n is getting very large—but the amount, A, is going toward a fixed number. This fixed number is the value of the continuously compounded interest where $m = \infty$.

Before you continue, you should verify my arithmetic—but do not verify the entire table, as there are eleven separate lines! Let's verify the daily one together. As we said before, bankers believe that there are 360 days per year. Why they have this delusion is unknown to me, because you would think they'd know that a year has 365 or 366 days, depending on if it is a leap year or not. In any case, that means a daily compounded loan has $m = 360$. Next, we know that $i = r/m$ and since $r = 0.05$ in this case, our calculator tells us that $i = 0.05/360 = 0.0001388\overline{8}\cdots$. The principal is given to us as \$ 1,000,000.00. All we need now is n, and $n = m \times t = (360)(20) = 7200$. Finally, we have

$$A = P(1+i)^n = 1,000,000(1 + 0.00013888\cdots)^{7200} = (1,000,000)(2.71809\cdots) = 2,718,093.08$$

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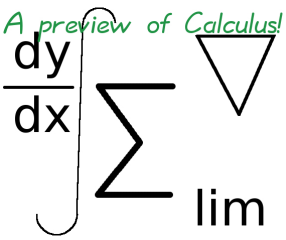
The phenomenon noted in the previous box is pretty intuitive. As you start compounding more and more often, the amount (value) of the loan keeps increasing. As there isn't much difference between a second and a minute, over the time span of 20 years, it also is intuitive that the last few rows should be identical. That's because the number of compounding periods per year is "almost infinity," and so either the mathematical behavior should go to a fixed number (the limit) or go crazy (this is called divergence).



In calculus, the concept of "a limit" allows us to make the previous idea rigorous. A person with even a light familiarity with calculus would say:

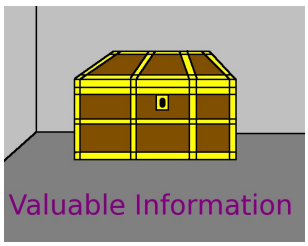
- As m goes to infinity, i goes to 0, or
$$\lim_{m\to\infty} i = 0$$
- As m goes to infinity, n goes to infinity, or
$$\lim_{m\to\infty} n = \infty$$
- As m goes to infinity, A goes to $2,718,281.\cdots$ or
$$\lim_{m\to\infty} A = 2,718,281.\cdots$$

. . . and this is one of the many reasons why any person interested in business should learn some calculus, no matter how little: calculus helps us learn to deal with the infinite and the infinitesimal— two things our brains have no ability to comprehend naturally.



This number, $2.718281828\cdots$ is so special that it gets a name, "e," just like $3.14159265\cdots$ is called π. This number was popularized by Leonhard Euler (1707–1783), but was discovered by Jacob Bernoulli (1654–1705). We'll have cause to talk about Jacob Bernoulli later, so we'll talk about Euler now.

Euler wrote a lot; his collected works would fill roughly 60 books, and he holds the record for the total page-count of any publishing mathematician. Nearly everything Euler wrote was newly discovered by him. His contributions included optics, parts of calculus, astronomy, and the famous Königsburg Bridges problem, which has tickled math enthusiasts for a little under 300 years. He was known to have had numerous conversations with world leaders like Peter the Great and Catherine the Great of Russia, as well as Frederick the Great of Prussia.



The formula for continuously compounded interest is given by

$$A = Pe^{rt}$$

As usual, A is the amount, P is the principal, r is the interest rate per year, and t is time, in years.

One should never assume that interest is compounded continuously unless the problem expressly says so. Some high finance uses continuous compounding, and I am told that some credit cards and savings accounts use it too, but mainly it is used as an approximation to other types of interest to make the math easier. The e^x function is particularly suitable in calculus.

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For Example :

Suppose someone deposits $ 50,000 compounded at 7% continuously, for 5 years. How much is the amount at the end of the term? What is the total interest earned?

We start with

$$A = Pe^{rt} = (50,000)e^{0.07\times 5} = (50,000)(1.41906\cdots) = 70,953.37$$

and the total interest is

$$70,953.37 - 50,000 = 20,953.37$$

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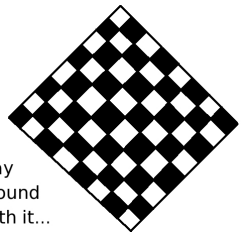


Repeat the above problem for 6% and 8%. [Answer: For 6% the amount is $ 67,492.94 and the total interest is $ 17,492.94. For 8% the amount is $ 74,591.23 and the total interest is $ 24,591.23.]

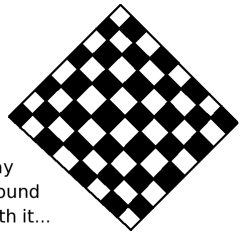
3-10-3

For Example :

Suppose that after 4 years of continuous compound interest, at the rate of 6%, an account has $ 54,321 in it. How much was in it originally?

$$\begin{aligned} A &= Pe^{rt} \\ 54,321 &= Pe^{4\times 0.06} \\ 54,321 &= Pe^{0.24} \\ 54,321 &= P(1.27124\cdots) \\ \frac{54,321}{1.27124\cdots} &= P \\ 42,730.41 &= P \end{aligned}$$

3-10-4



How about if it were 5%? Or 8%? [Answer: For 5% it would have been $ 44,474.27, and for 8% it would have been $ 39,445.14.]

3-10-5

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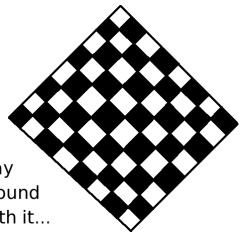
If you have $x = 10^y$, you've learned by now that you can use the common logarithm to obtain $\log x = y$. You might wonder what to do when you have $x = e^y$. The *natural logarithm* (abbreviated as ln) works just like the common logarithm, except that it is built off e instead of 10. Just as

$$\log 10 = 1, \qquad \log 100 = 2, \qquad \log 1000 = 3, \qquad \log 10,000 = 4$$

it is likewise true that

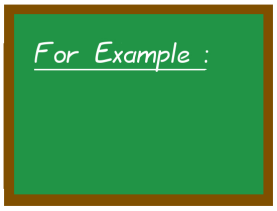
$$\ln e = 1, \qquad \ln e^2 = 2, \qquad \ln e^3 = 3, \qquad \ln e^4 = 4$$

and so on. As you can see, the natural logarithm is the inverse function of e^x. Otherwise, the two logarithms are identical. All of the normal laws of logarithms apply to $\ln x$ just as they did to $\log x$.



3-10-6

- Evaluate $\ln e^6$. [Answer: 6.]
- Evaluate $\ln e^x$. [Answer: x.]
- Evaluate $\ln 6x$. [Answer: $(\ln 6) + (\ln x)$.]
- Evaluate $\ln(x^2y^3)$. [Answer: $2(\ln x) + 3(\ln y)$.]

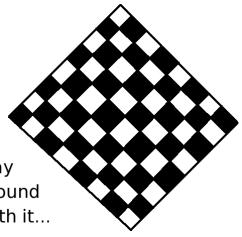


3-10-7

Let's try using our new tool. Suppose an account grows from \$ 21,000 to \$ 25,000 in 5 years when continuously compounded. What was the rate?

$$\begin{aligned} A &= Pe^{rt} \\ 25,000 &= (21,000)e^{5r} \\ 1.19047\cdots &= e^{5r} \\ \ln 1.19047\cdots &= \ln e^{5r} \\ \ln 1.19047\cdots &= 5r \\ 0.174353\cdots &= 5r \\ 0.0348706\cdots &= r \end{aligned}$$

and thus $r = 3.48707\cdots\%$.



3-10-8

- Repeat the previous example using 4 years. [Answer: $4.35883\cdots\%$.]
- Repeat the previous example using 6 years. [Answer: $2.90588\cdots\%$.]

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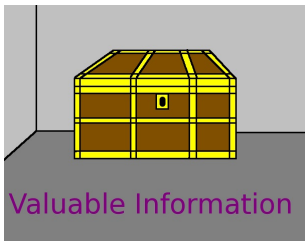
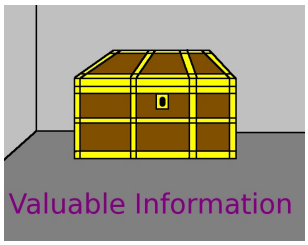


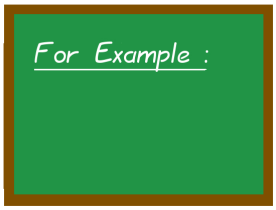
Mathematics has many maneuvers that can be considered move & counter-move. For example, addition and subtraction are opposites, squaring and square-rooting are opposites, and multiplying and dividing are opposites. We are now exploring that exponentiating is the opposite of the logarithm. This can be summarized by the following list, an expansion of what was found on Page 28, Page 349 and Page 505. This important topic is called "the theory of inverse functions."

- If $4x = 64$ and you want to "undo" the "times 4," you do $64/4$ to learn $x = 16$.
- If $x/2 = 64$ and you want to "undo" the "divide by 2," you do 64×2 to learn $x = 128$.
- If $x + 13 = 64$ and you want to "undo" the "plus 13," you do $64 - 13$ to learn $x = 51$.
- If $x - 12 = 64$ and you want to "undo" the "minus 12," you do $64 + 12$ to learn $x = 76$.
- If $x^2 = 64$ and you want to "undo" the "square," you do $\sqrt{64}$ to learn $x = 8$.
- If $x^3 = 64$ and you want to "undo" the "cube," you do $\sqrt[3]{64}$ to learn $x = 4$.
- If $x^6 = 64$ and you want to "undo" the "sixth power," you do $\sqrt[6]{64}$ to learn $x = 2$.
- If $10^x = 64$ and you want to "undo" the "ten to the," you do $\log 64$ to learn $x = 1.80617\cdots$.
- If $\log x = 64$ and you want to "undo" the "logarithm," you do 10^{64} to learn $x = 10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000$.
- If $e^x = 64$ and you want to "undo" the "e to the," you do $\ln 64$ to learn $x = 4.15888\cdots$.
- If $\ln x = 64$ and you want to "undo" the "logarithm," you do e^{64} to learn $x \approx 6.23514\cdots \times 10^{27}$, where $\approx$ means "approximately equals."



One way of remembering this relationship is

- Just as $10^x = junk$ implies $x = \log junk$, so does $e^x = junk$ imply $x = \ln junk$.
- Just as $\log x = junk$ implies $x = 10^{junk}$, so does $\ln x = junk$ imply $x = e^{junk}$.
- Other than that, $\log x$ and $\ln x$ operate identically.



3-10-9

Consider the following

$$100 = e^{12t-5}$$

This at first looks really worrisome. Now we have the advantage of knowing that e^x and $\ln x$ are inverse functions. That means that the $\ln x$ function can undo the e^x function. So we can take $\ln x$ of both sides.

$$\begin{aligned} 100 &= e^{12t-5} \\ \ln 100 &= \ln e^{12t-5} \\ \ln 100 &= 12t - 5 \\ 4.60517\cdots &= 12t - 5 \\ 9.60517\cdots &= 12t \\ \frac{9.60517\cdots}{12} &= t \\ 0.800430\cdots &= t \end{aligned}$$

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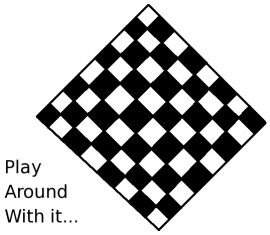
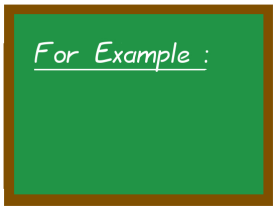


3-10-10

Try the following:

- Solve $75 = 5e^{13t+8}$ for t. [Answer: $t = -0.407073\cdots$.]
- Solve $180 = 4e^{-10t-6}$ for t. [Answer: $t = -0.980666\cdots$.]
- Solve $209 = 6e^{17t+4}$ for t. [Answer: $t = -0.0264367\cdots$.]
- Solve $43 = 3e^{-2t-1}$ for t. [Answer: $t = -1.83129\cdots$.]

Here's a useful hint: when you want to get e on your calculator, press 1 and then press e^x. This works because $e^1 = e$. Alternatively, on some calculators, you press 1 after the e^x button. Lastly, some calculators have a button just for e.



3-10-11

Suppose we know that

$$4 + 15\ln x = 12$$

Then naturally

$$\begin{aligned} 15\ln x &= 8 \\ \ln x &= 8/15 \\ x &= e^{8/15} \\ x &= 1.70460\cdots \end{aligned}$$

We can check this by plugging x into the original equation.

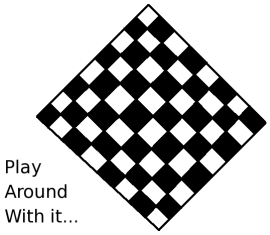


Always plug your answer back into the original question, to make sure it works!

In this case, we obtain

$$4 + 15\ln 1.70460 = 4 + 15(0.533330\cdots) = 11.9999\cdots$$

which is a perfect match!



3-10-12

Try solving the following:

- Solve for x in $12 + 30\ln 2x = 200$. [Answer: $263.359\cdots$.]
- Solve for x in $36 + 20\ln x = 200$. [Answer: $3640.95\cdots$.]
- Solve for x in $24 + 40\ln 4x = 200$. [Answer: $20.3627\cdots$.]
- Solve for x in $48 + 10\ln 3x = 200$. [Answer: $1.33092\cdots \times 10^6$.]

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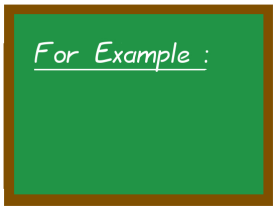
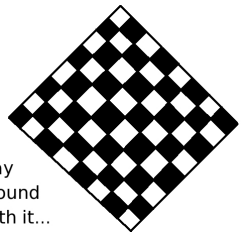


3-10-13

Suppose someone has a credit card that compounds continuously. How long will it take, at 19.95% interest, for $ 20,000 in debt to become $ 50,000?

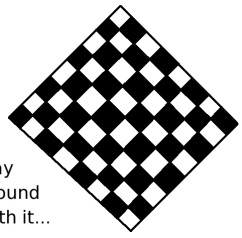
$$\begin{aligned} A &= Pe^{rt} \\ 50,000 &= (20,000)e^{0.1995t} \\ 2.50 &= e^{0.1995t} \\ \ln 2.50 &= 0.1995t \\ 0.916290\cdots &= 0.1995t \\ 4.59293\cdots &= t \end{aligned}$$

At this (exorbitant) rate of interest, a mere 4.59 years, or roughly 4 years and 7 months, would be required to cause the $ 20,000 debt to balloon to $ 50,000.



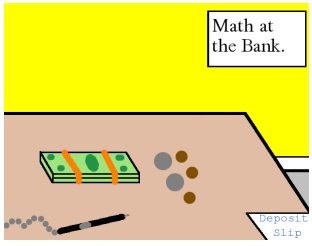
3-10-14

How about if, in the previous example, the interest rate were 24.95% or 29.95%? [Answer: For 24.95% it would be 3.67250 years, and for 29.95% it would be 3.05940 years.]

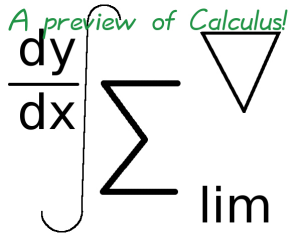


3-10-15

- If a debt of $ 20,000 grows to $ 45,000 in only 4 years, what rate of continous compound interest was used? [Answer: $20.2732\cdots$%.]
- And if it were in 5 years? [Answer: $16.2186\cdots$%.]



One would be very unfortunate to have such an interest rate. Sadly, 19.95% is not at all uncommon as an interest rate among credit cards. Also, note that the credit card company can change your interest rate if they so desire. We examined that on Page 515, when we learned what it means to *go into default*. I earnestly hope that you will always approach credit cards with a healthy dose of fear.



There are many ways to give the formal definition of e. It does not make too much sense to just say $e = 2.718281828\cdots$, particularly because we cannot write down all of the infinitely many digits without an infinitely large piece of paper. So how will we define it?

Two definitions come form calculus, and one comes from the limit of the compound interest formula (we saw that earlier). Each definition is equivalent to the other, though it might take a great deal of mathematics to get from one to the other. Usually more advanced books pick one definition, and derive the others as "properties" of e.

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For Example :

3-10-16

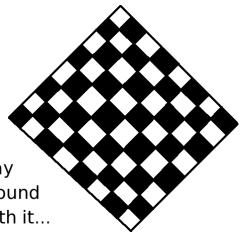
Jacob Bernoulli's definition of e is derived from compound interest. If we let the interest rate be 100% (just for the sake of calculation), compounded m times a year, then what is the compounding factor, in one year? Well, recall the way to find the compounding factor is to use the compound interest formula and set $P = 1$. Of course, $i = r/m = 1/m$ in this case, and likewise $n = mt = m1 = m$ So we'd have

$$A = P(1+i)^n = (1)(1+1/n)^n$$

We could ask ourselves, with $n = 10,000$, what is A? Although Bernoulli was thinking of $n = \infty$, on the other hand $10,000$ is a big number, so this should be a good approximation. We get

$$A = (1+1/n)^n = (1+1/10,000)^{10,000} = (1.0001)^{10,000} = 2.718145926825225\cdots$$

The residual error of this approximation turns out to be $-1.35901\cdots \times 10^{-4}$ and the relative error turns out to be $-4.99954\cdots \times 10^{-5}$, both quite excellent.



3-10-17

What if I were to try

- Using $n = 1000$, what is the approximate value of e stated? What is the residual error? The relative error? [Answer: The value is $2.71692\cdots$, which has an residual error of $-0.00135789\cdots$, and a relative error of -4.99542×10^{-4}.]
- Using $n = 100,000$, what is the approximate value of e stated? What is the residual error? The relative error? [Answer: The value would be $2.71826\cdots$, which has an residual error of -1.35912×10^{-5}, and a relative error of -4.99995×10^{-6}.]

For Example :

3-10-18

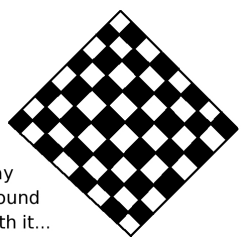
An alternative definition of e is the following infinite sum:

$$e = 1 + \frac{1}{(1)} + \frac{1}{(1)(2)} + \frac{1}{(1)(2)(3)} + \frac{1}{(1)(2)(3)(4)} + \frac{1}{(1)(2)(3)(4)(5)} + \frac{1}{(1)(2)(3)(4)(5)(6)} + \cdots$$

So let's add up those first seven terms, and see how close we get! We have

$$e = 1 + 1 + 0.5 + 0.16\overline{6} + 0.0416\overline{6} + 0.00833\overline{3} + 0.001388\overline{8} + \cdots = 2.7180\overline{5}$$

and the residual error turns out to be $-2.26272\cdots \times 10^{-4}$ and the relative error is $-8.32411\cdots \times 10^{-5}$, which is quite excellent.



3-10-19

What if I do

- ...eight terms of that sum which appeared in the previous box? What is the approximate value of e stated? What is the residual error? The relative error? [Answer: The approximation is $2.71825\cdots$, the residual error is $-2.78602\cdots \times 10^{-5}$, and the relative error is $-1.02491\cdots \times 10^{-5}$, or $0.00102\cdots\%$, very tiny indeed.]
- ...six terms of that sum which appeared in the previous box? What is the approximate value of e stated? What is the residual error? The relative error? [Answer: The approximation is $2.716\overline{6}$, the residual error is $-1.61516\cdots \times 10^{-3}$, and the relative error is $-5.94184\cdots \times 10^{-4}$, or $0.0594\cdots\%$, not bad at all.]

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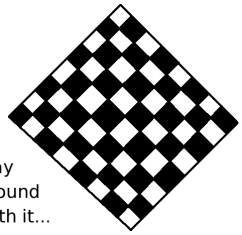
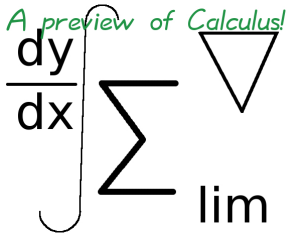


3-10-20

Calculate the sum

$$1 - \frac{1}{(1)} + \frac{1}{(1)(2)} - \frac{1}{(1)(2)(3)} + \frac{1}{(1)(2)(3)(4)} - \frac{1}{(1)(2)(3)(4)(5)} + \frac{1}{(1)(2)(3)(4)(5)(6)}$$

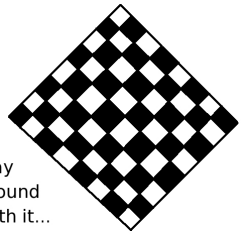
and write down the grand total. This comes to an approximation for $1/e$, which I think is neat. What is the residual error and relative error compared to the actual $1/e$? [Answer: $0.368055\overline{5}$; residual error $= 1.76114 \times 10^{-4}$; Relative Error $= 4.78728 \times 10^{-4}$.]



There are many kinds of sequences and series (long sums of terms in a progression) in mathematics. There is the geometric series or geometric progression, which will be very important to our work in finance, and that is explained on Page 557. There is also the arithmetic series or arithmetic progression, which will be explained on Page 579. These two are the only series-types covered in this book.

In calculus, the harmonic series and the Taylor series are very important, and the previous box's approximation of e is an example of a Taylor series. There's also the Laurent Series, which often comes up in advanced courses. Finally, in scientific computation (my speciality) the Fourier series is extraordinarily useful, and forms the mathematical backbone of signal processing—whether working with sound or images.

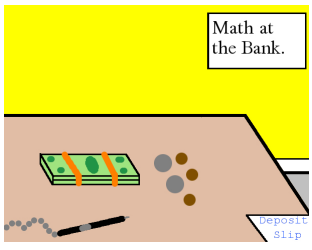
After that exploration into pure mathematics, I'm sure you'd like to return to finance.



3-10-21

Let's say that Bob has access to an account paying 7% compounded continuously, and he has $ 21,000 deposited there.

- How long will it take for his wealth to double? [Answer: $9.90210\cdots$ years.]
- If Charlie is less wealthy, and has only $ 3000 in the bank, but uses the same type of account, how long would it take for his wealth to double? [Answer: $9.90210\cdots$ years.]
- If Alice is an investment banker and has $ 49,000 in the bank, but uses the same type of account, how long would it take for her wealth to double? [Answer: $9.90210\cdots$ years.]



The results shown in the previous box are not a coincidence. The principal doesn't matter. Doubling time is a property of the investment itself and doesn't care about the principal. That's because in each case we reach the equation

$$2 = e^{rt}$$

and that equation has no P nor any A in it.

We'll see how to compute the doubling time of forms of compound interest that are not continuously compounded, such as monthly or quarterly, in the module "Solving Problems using Logarithms," starting on Page 534.

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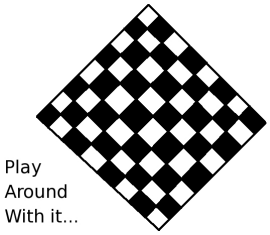
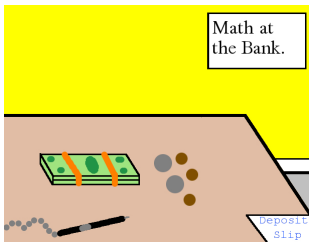


3-10-22

- What is the doubling time for 6% compounded continuously? [Answer: $11.5524\cdots$ years.]
- What is the doubling time for 5% compounded continuously? [Answer: $13.8629\cdots$ years.]



There's a system of notation you might see some bankers use

- The notation 3%(2) means 3% compounded semi-annually.
- The notation 4%(12) means 4% compounded monthly.
- The notation 5%(4) means 5% compounded quarterly.
- The notation 6%(∞) means 6% compounded continuously.
- The notation $x\%(y)$ means $x\%$ compounded $1/y$ times per year, for any positive real number y and any real number x.



There is a cute and simple formula to find the annual percentage yield (APY) or annual effective rate (AER) of a continuously compounded loan or investment. However, it only works without fees.

The APY or AER (without fees) when $A = Pe^{rt}$ is just $e^r - 1$. I will show you how to do compute the APY or AER with fees in the next box.

Nonetheless, it is often best not to memorize such shortcuts and instead use the technique in the following box, which works for continuous compounded interest just as it did for ordinary compound interest on Page 408, in the module "Predatory Lending and the AER." After all, if you memorize a formula, you might memorize it incorrectly. The technique in the next box works both with and without fees.

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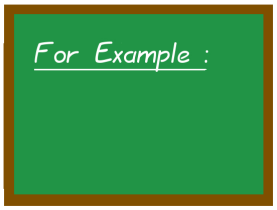


3-10-23

The APY (with and without fees) for continuous compound interest is computed the same way that it is for the other types of compound interest.

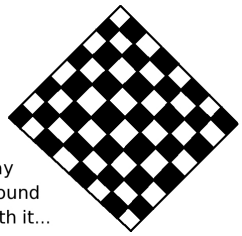
Suppose you have a loan for \$ 12,000 compounded continuously at 19.95%, and that there is also \$ 300 in fees. This loan is for two years. The amount without fees would be

$$A = Pe^{rt} = (12,000)e^{0.1995\times 2} = (12,000)(1.49033\cdots) = 17,884.00$$

and with fees thus $17,884 + 300 = 18,184.00$. Next, for the APY, one merely must calculate what rate, compounded *annually* and in the same length of time (two years), would produce the same amount, with or without fees, from the principal. In each case we have

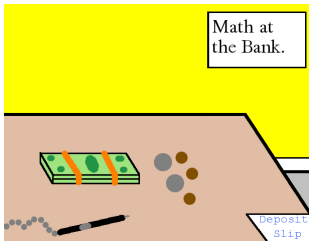
Without Fees			With Fees		
A	$=$	$P(1+i)^n$	A	$=$	$P(1+i)^n$
$17,884.00$	$=$	$(12,000)(1+i)^2$	$18,184.00$	$=$	$(12,000)(1+i)^2$
$17,884.00/12,000$	$=$	$(1+i)^2$	$18,184/12,000$	$=$	$(1+i)^2$
$1.4903\overline{3}$	$=$	$(1+i)^2$	$1.5153\overline{3}$	$=$	$(1+i)^2$
$\sqrt{1.4903\overline{3}}$	$=$	$(1+i)^2$	$\sqrt{1.5153\overline{3}}$	$=$	$(1+i)^2$
$1.22079\cdots$	$=$	$1+i$	$1.23098\cdots$	$=$	$1+i$
$0.220792\cdots$	$=$	i	$0.230988\cdots$	$=$	i

and so the APY-without-fees is 22.07% and the APY-with-fees is 23.09%.

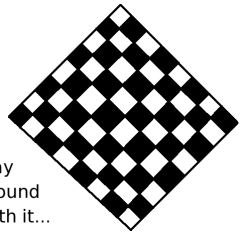


3-10-24

- What if the fee were \$ 3000, what would the APY (with and without) become? [Answer: The APY-with-fees is 31.92% and of course the APY-without-fees is unchanged at 22.07%, because it doesn't know about fees.]
- Recall, the *finance charge* is the total interest paid, which is $A - P$, for any loan.
- In the previous loan, with the fees being \$ 3000, what is the finance charge? And if we change the fees back to \$ 300? [Answer: \$ 8884 and \$ 6184.]



Right before the Savings & Loan Crisis of the late 1980s, the government limited banks to 5% interest on savings accounts and similar instruments, while S&L's were limited to 5.50%. One way to "bend" the upper limit was to offer the 5.50%, but compound it continuously, which made it a little bit higher of a rate than if compounded annually. Since the personal computer was also becoming popular and cheap at that time, the added computational effort was not really a barrier. Let's see how much of an edge that provided.



3-10-25

Suppose one S&L gives 5.50% compounded annually. Their competitor gives 5.50% compounded continuously.

- What is the APY in the annual case? [Answer: 5.50%.]
- What is the APY in the second case? [Answer: $5.65406\cdots$%.]
- That's not much of an edge, but non-trivial. Suppose one invests \$ 10,000 for 5 years. What is the total interest earned in the annual case? [Answer: \$ 3069.60.]
- What is the total interest earned in the continuous case? [Answer: \$ 3165.30.]

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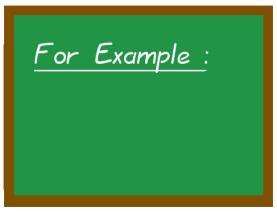
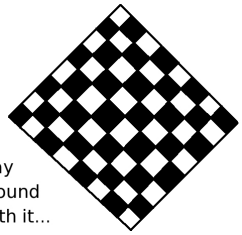


You were told that the compounding factor can be found by setting $P = 1$ and the cost per thousand by setting $P = 1000$. (See Page 430.) This remains true. Let's consider 29.95% compounded continuously for a 2 year period. We'd have

$$A = Pe^{rt} = (1000)(e^{0.2995\times 2}) = 1000e^{0.599} = 1820.29$$

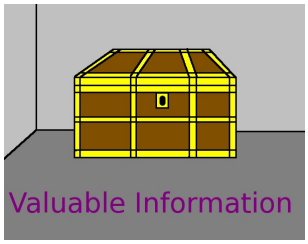
and so the cost per thousand is 1820.29 and the compounding factor is $1.82029\cdots$.

3-10-26



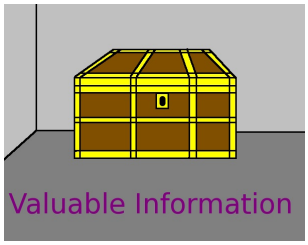
What are the compounding factors and cost per thousand for 19.95%? or for 24.95%, compounded continuously for one year? [Answer: The compounding factors are $1.22079\cdots$ and $1.28338\cdots$ respectively, and therefore the costs per thousand are \$ 1220.79 and \$ 1283.38.]

3-10-27



A common question is which logarithm should a student use?

- If the problem explicitly requires the use of a particular logarithm, definitely do not disobey that. Your instructor would be upset if you disobey.
- If the problem already contains e in some form, then use the natural logarithm.
- If not, and the problem has 10^x in some way, then use the common logarithm.
- If the numbers in the problem are large enough or small enough to be in scientific notation, then use the common logarithm. Remember, the part of the common logarithm to the left of the decimal point in a positive number is its exponent in scientific notation.
- Otherwise, it does not matter. You should practice using both.

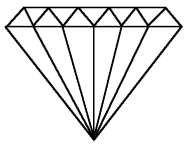


Sometimes we'll raise e to very intricate powers. Examples of this could look like:

$$e^{2x^2+3x+4+\sqrt{5}}$$

which can be much easier written as $exp(2x^2 + 3x + 4 + \sqrt{5})$. This abbreviation works as follows: $exp(x)$ means e^x. The idea is that even though we are using three letters instead of one, we are also saving the reader a headache, as the long expression $2x^2 + 3x + 4 + \sqrt{5}$ can be written at a normal font size.

Hard but Valuable!



There is another formula relating to logarithms that can be useful for time to time. It is the "change of base formula" and helps converting one type of logarithm into another. You can convert among the natural logarithm, the common logarithm, or the logarithm with any particular base you need.

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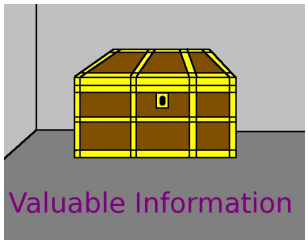

In my experience, it is a recipe for disaster to mix the common and natural logarithms together in one problem. Luckily, the change-of-base formula can help to extract you from such a mess. It turns out that

$$\log x = \frac{\ln x}{\ln 10}$$

and also

$$\ln x = \frac{\log x}{\log e}$$

The "general form" of the change of base formula is

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

and can be used to turn logarithms with exotic bases into logarithms that you can compute with a hand-held calculator. If you plug $b = 10$ or $b = e$ into the above "general form" then you will get the two more compact forms which were found earlier in this box.

Now, over three boxes, we'll explore where the "change of base" formula comes from.



Suppose $y = \log x$. This is the same thing as saying $10^y = x$. Let's see where we can take that

$$\begin{aligned} 10^y &= x \\ \ln 10^y &= \ln x \\ y \ln 10 &= \ln x \\ y &= \frac{\ln x}{\ln 10} \\ \log x &= \frac{\ln x}{\ln 10} \end{aligned}$$



To go the other way, if $y = \ln x$ then that's the same thing as saying $e^y = x$. Then we can do the same process

$$\begin{aligned} e^y &= x \\ \log e^y &= \log x \\ y \log e &= \log x \\ y &= \frac{\log x}{\log e} \\ \ln x &= \frac{\log x}{\log e} \end{aligned}$$

and we get the desired result.

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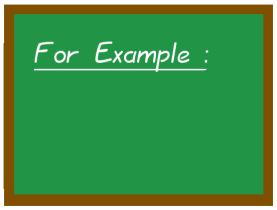
Now for the general form, if $y = \log_b x$ then that means $b^y = x$. We can proceed:

$$\begin{aligned} b^y &= x \\ \ln b^y &= \ln x \\ y \ln b &= \ln x \\ y &= (\ln x)/(\ln b) \\ \log_b x &= (\ln x)/(\ln b) \end{aligned}$$

Or the exact same proof can be written by replacing the natural logarithm $\ln x$ with the common logarithm $\log x$.



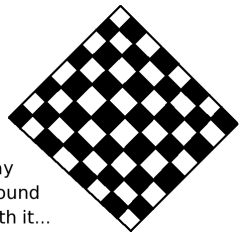
How would you use the change of base formula to compute $\log_5 625$, using the common logarithm button on your calculator?

$$\log_5 625 = \frac{\log 625}{\log 5} = \frac{2.79588001\cdots}{0.698970004\cdots} = 4$$

Next, we'd check that computation with

$$5^4 = 625$$

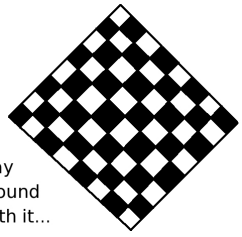
3-10-28



How would you compute $\log_7 343$ using the common logarithm button on your calculator?

- What formula would you use? [Answer: $\log_7 343 = (\log 343)/(\log 7)$.]
- What is the final answer? [Answer: $\log_7 343 = 3$.]
- How would you check it? [Answer: $7^3 = 343$.]

3-10-29



How would you compute $\log_2 4096$ using the natural logarithm button on your calculator?

- What formula would you use? [Answer: $\log_2 4096 = (\ln 4096)/(\ln 2)$.]
- What is the final answer? [Answer: $\log_2 4096 = 12$.]
- How would you check it? [Answer: $2^{12} = 4096$.]

3-10-30

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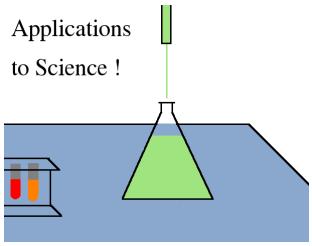
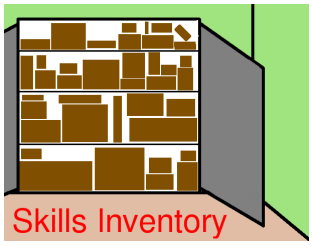


In my life, the change-of-base formula comes up often because computer scientists find it extremely convenient to use the logarithm "base 2," and no hand-held calculators have a button for that! Since I work in the intersection of computer science and mathematics, I frequently need to calculate the logarithm base 2.

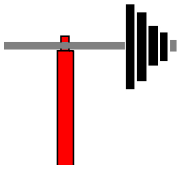
My research area is cryptography (the science of codes) and the logarithm base two helps figure out how much time it would take for a computer to guess all the combinations possible to break a code. This is a very primitive way of breaking a code, but it is also easy and it is how hobby hackers might attempt to break codes used on the internet to protect private information. So it is crucial that this be calculated, so that we can insure that such a primitive attempt is rendered infeasible.



We have learned the following skills in this module:

- To calculate continuous compound interest.
- To calculate the finance charge, total interest paid, compounding factor and cost-per-thousand of continuous compound interest.
- To identify continuous compound interest as the limit of ordinary compound interest when $m \to \infty$, and to compare and contrast continuous compound interest with very high numerical values of m.
- To calculate the AER or APY in a continuous compound interest environment.
- To use the natural logarithm to solve exponential equations in one variable.
- To identify which logarithm should be used in a given problem.
- Optionally, to use the change-of-base formula to switch between common logarithms and natural logarithms.
- As well as the vocabulary terms: continuous compounding, doubling time, finance charge, going into default, and natural logarithm.

Try some Exercises!



Coming soon!

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