An expressive two-sorted spatial logic for plane projective geometry

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ABSTRACT. The aim of this paper is to introduce an expressive two-sorted spatial logic for plane projective geometry. We firstly attack the problem of the definability of multifarious classes of incidence frames and the problem of the reductions between several sublanguages of our language. We secondly attack the problem of the axiomatization of the logics of some natural classes of incidence frames.

1 Introduction

Projective planes are one of the most classical relational structures in geometry. They consist of either a nonempty set of points and a ternary relation of collinearity or a nonempty set of points, a nonempty set of lines and a binary relation of incidence. In the latter case any two distinct points are incident with precisely one common line and any two distinct lines are incident with exactly one common point. In general it is also assumed that any point is incident with at least two lines and any line is incident with at least two points. See Beutelspacher [6] for a survey. The aim of this paper is to introduce an expressive two-sorted spatial logic for plane projective geometry.

A modal treatment of plane projective geometry has been considered for the first time by Balbiani [1] and Venema [18] who interpreted the spatial modalities $[I]$ and $[O]$ by the incidence relation between points and lines, $[I]a$ being read "in all lines x on the current point, a is realized at x" and $[O]A$ being read "in all points X in the current line, A is realized at X". In this paper we consider the new spatial modalities $[01]$ and $[10]$, $[01](a,b)$ being read "in all lines x, y on the current point, if x and y are different then a is realized at x or b is realized at y" and $[10](A,B)$ being read "in all points X, Y in the current line, if X and Y are different then A is realized at X or B is realized at Y".

The paper has three major sections. Section 2 introduces our language based on $[01]$ and $[10]$ and interprets it in projective planes. Section 3 attacks the problem of the definability of multifarious classes of incidence frames and the problem of the reductions between several sublanguages of our language. Section 4 attacks the problem of the axiomatization of the logics of some natural classes of incidence frames. In the annex we prove some of our results. We assume the reader is at home with tools and techniques in modal logic such as rules for the undefinable and the canonical model construction. For more on these see Blackburn et al [7], Chagrov and Zakharyaschev [8] or Kracht [13].

2 Syntax and semantics

2.1 Syntax

Now it is time to present the language we will be working with. Let PV be a countable set of *point variables*, with typical members denoted P, Q, etc, and LV be a countable set of *line variables*, with typical members denoted p, q, etc. We define the set of all *point formulas*, with typical members denoted A, B, etc, and the set of all *line formulas*, with typical members denoted a, b, etc, by the rules

- $A := P \mid \bot \mid \neg A \mid (A \vee B) \mid [01](a, b)$ and
- $a := p \mid \bot \mid \neg a \mid (a \vee b) \mid [10](A, B)$.

We adopt the standard rules for omission of the parentheses. Point formulas and line formulas constitute the formulas of our language, with typical representatives denoted ϕ, ψ, etc. The point formula $[01](a, b)$ means "in all lines x, y on the current point, if x and y are different then a is realized at x or b is realized at y" and the line formula $[10](A, B)$ means "in all points X, Y in the current line, if X and Y are different then A is realized at X or B is realized at Y". We make use of the abbreviations

- $[\in]a := [01](a, a)$, i.e. "at the very most, one line on the current point does not realize a",
- $[I]a := [01](a, \bot)$, i.e. "in all lines on the current point, a",
- $[\cdot]A := [01]([10](A, \bot), \bot)$, i.e. "in all points, A",
- $\odot A := \neg A \wedge [01]([10](A, A), \bot)$, i.e. "in all points but the current one, A",
- $[\ni]A := [10](A, A)$, i.e. "at the very most, one point in the current line does not realize A",
- $[O]A := [10](A, \bot)$, i.e. "in all points in the current line, A",
- $[-]a := [10]([01](a, \bot), \bot)$ i.e. "in all lines, a", and
- $\ominus a := \neg a \wedge [10]([01](a, a), \bot)$ i.e. "in all lines but the current one, a".

Remark that $[\cdot]A$ is $[I][O]A$, $\odot A$ is $\neg A \wedge [I][\ni]A$, $[-]a$ is $[O][I]a$ and $\ominus a$ is $\neg a \wedge [O][\in]a$. The other constructs are defined as usual. In particular we have dual operators for our boxes which are defined by $\langle 01\rangle(a, b) := \neg[01](\neg a, \neg b)$, $\langle\in\rangle a := \neg[\in]\neg a$, $\langle I\rangle a := \neg[I]\neg a$, $\langle\cdot\rangle A := \neg[\cdot]\neg A$, $\langle 10\rangle(A, B) := \neg[10](\neg A, \neg B)$, $\langle\ni\rangle A := \neg[\ni]\neg A$, $\langle O\rangle A := \neg[O]\neg A$ and $\langle -\rangle a := \neg[-]\neg a$.

2.2 Semantics

Now it is time to interpret our language in relational structures. An *incidence frame* is a relational structure

- $\mathcal{F} = (Po, Li, I, O)$

where Po is a nonempty set of "points", with typical members denoted X, Y, etc, Li is a nonempty set of "lines", with typical members denoted x, y, etc, I is a binary relation between points and lines and O is a binary relation between lines and points. If $I(X, x)$, i.e. the relation I holds for point X and line x, we say that "X is in x" and if $O(x, X)$, i.e. the relation O holds for line x and point X, we say that "x is on X". We introduce special notations for the ternary relations

- $In(X, y, z) := (I(X, y) \wedge I(X, z) \wedge y \neq z)$ and
- $On(x, Y, Z) := (O(x, Y) \wedge O(x, Z) \wedge Y \neq Z)$

which will be used further. $\mathcal{F}$ is said to be *rooted* if

- $\exists X((I \circ O)^\star(X) = Po \wedge (O \circ I)^\star(I(X)) = Li)$ or
- $\exists x((I \circ O)^\star(O(x)) = Po \wedge (O \circ I)^\star(x) = Li)$

where $I(X) = \{x\colon\ I(X, x)\}$, $O(x) = \{X\colon\ O(x, X)\}$, $\circ$ denotes composition and * denotes iteration. For all natural integers n such that $n \geq 2$, we shall say that $\mathcal{F}$ is *n-rich* if

- $\forall X(Card(I(X)) \geq n)$ and
- $\forall x(Card(O(x)) \geq n)$.

$\mathcal{F}$ is said to be *ω-rich* if for all natural integers n such that $n \geq 2$, $\mathcal{F}$ is n-rich. We shall say that $\mathcal{F}$ is *correct* if

- $\forall X \forall x(I(X, x) \rightarrow O(x, X))$ and
- $\forall x \forall X(O(x, X) \rightarrow I(X, x))$.

$\mathcal{F}$ is said to be *projective* if

- $\forall X \forall Y(I(X) \cap I(Y) \neq \emptyset)$ and
- $\forall x \forall y(O(x) \cap O(y) \neq \emptyset)$.

We shall say that $\mathcal{F}$ is *normal* if

- $\forall X \forall Y \forall z \forall t(I(X, z) \wedge I(Y, z) \wedge I(X, t) \wedge I(Y, t) \rightarrow X = Y \vee z = t)$ and
- $\forall x \forall y \forall Z \forall T(O(x, Z) \wedge O(y, Z) \wedge O(x, T) \wedge O(y, T) \rightarrow x = y \vee Z = T)$.

$\mathcal{F}$ is said to be *Arguesian* if, as a projective plane, it satisfies the theorem of Desargues. See [6] at page 109 for details. We shall say that $\mathcal{F}$ is *Pappian* if, as a projective plane, it satisfies the theorem of Pappus. See [6] at page 117 for details. An *incidence model* based on $\mathcal{F}$ is an ordered triple

- $\mathcal{M} = (\mathcal{F}, \pi, \lambda)$

where π is a function assigning to each point variable $P \in PV$ a subset $\pi(P)$ of Po and λ is a function assigning to each line variable $p \in LV$ a subset $\lambda(p)$ of Li. We define the notion of a point formula A being *true* at a point $X \in Po$, in symbols $\mathcal{M}, X \models A$, and the notion of a line formula a being *true* at a line $x \in Li$, in symbols $\mathcal{M}, x \models a$, by

- $\mathcal{M}, X \models P$ iff $X \in \pi(P)$,
- $\mathcal{M}, X \not\models \bot$,
- $\mathcal{M}, X \models \neg A$ iff $\mathcal{M}, X \not\models A$,
- $\mathcal{M}, X \models A \vee B$ iff $\mathcal{M}, X \models A$ or $\mathcal{M}, X \models B$,
- $\mathcal{M}, X \models [01](a, b)$ iff for all lines y, z, if $In(X, y, z)$ then $\mathcal{M}, y \models a$ or $\mathcal{M}, z \models b$,
- $\mathcal{M}, x \models p$ iff $x \in \lambda(p)$,
- $\mathcal{M}, x \not\models \bot$,
- $\mathcal{M}, x \models \neg a$ iff $\mathcal{M}, x \not\models a$,
- $\mathcal{M}, x \models a \vee b$ iff $\mathcal{M}, x \models a$ or $\mathcal{M}, x \models b$ and
- $\mathcal{M}, x \models [10](A, B)$ iff for all points Y, Z, if $On(x, Y, Z)$ then $\mathcal{M}, Y \models A$ or $\mathcal{M}, Z \models B$.

Remark that the interpretation of the point formula $[01](a, b)$ involves the relation of inequality between lines and the interpretation of the line formula $[10](A, B)$ involves the relation of inequality between points.

LEMMA 1. *Let*

- $SEP_0 := [01](p, q) \to [I](p \vee r) \vee [I](q \vee \neg r)$ *and*
- $SEP_1 := [10](P, Q) \to [O](P \vee R) \vee [O](Q \vee \neg R)$.

Then

- *for all points* $X \in Po$, $\mathcal{M}, X \models SEP_0$ *and*
- *for all lines* $x \in Li$, $\mathcal{M}, x \models SEP_1$.

Par abus de langage let $\pi(A) = \{X\colon \mathcal{M}, x \models A\}$ and $\lambda(a) = \{x\colon \mathcal{M}, x \models a\}$. Operators like $[\in]$ and $[\ni]$ remind one of operators that are sometimes called graded modalities [10], seeing that

LEMMA 2.

- *For all points* $X \in Po$, $\mathcal{M}, X \models [\in]a$ *iff there exists a line* x *such that* $I(X) \setminus \{x\} \subseteq \lambda(a)$ *and*

- *for all lines $s \in Li$, $\mathcal{M}, x \models [\ni]A$ iff iff there exists a point X such that $O(x) \setminus \{X\} \subseteq \pi(A)$.*

The operators $[I]$ and $[O]$ were introduced by Balbiani [1] and Venema [18] in the course of their work on spatial logics, seeing that

LEMMA 3. *If $\mathcal{M}$ is 2-rich then*

- *for all points $X \in Po$, $\mathcal{M}, X \models [I]a$ iff $I(X) \subseteq \lambda(a)$ and*
- *for all lines $x \in Li$, $\mathcal{M}, x \models [O]A$ iff $O(x) \subseteq \pi(A)$.*

Operators like $[\cdot]$ and $[-]$ remind one of operators that are sometimes called universal modalities [12], seeing that

LEMMA 4. *If $\mathcal{M}$ is 2-rich, correct and projective then*

- *for all points $X \in Po$, $\mathcal{M}, X \models [\cdot]A$ iff $\pi(A) = Po$ and*
- *for all lines $x \in Li$, $\mathcal{M}, x \models [-]a$ iff $\lambda(a) = Li$.*

Operators like $\odot$ and $\ominus$ remind one of operators that are sometimes called difference modalities [14], seeing that

LEMMA 5. *If $\mathcal{M}$ is 2-rich, correct and projective then*

- *for all points $X \in Po$, $\mathcal{M}, X \models \odot A$ iff $\pi(A) = Po \setminus \{X\}$ and*
- *for all lines $x \in Li$, $\mathcal{M}, x \models \ominus a$ iff $\lambda(a) = Li \setminus \{x\}$.*

We shall say that point formula A is *valid* in $\mathcal{M}$, in symbols $\mathcal{M} \models A$, if $\pi(A) = Po$ and line formula a is *valid* in $\mathcal{M}$, in symbols $\mathcal{M} \models a$, if $\lambda(a) = Li$. Now consider an incidence frame $\mathcal{F} = (Po, Li, I, O)$. A formula ϕ is said to be *valid* in $\mathcal{F}$, in symbols $\mathcal{F} \models \phi$, if ϕ is valid in all incidence models based on $\mathcal{F}$. We shall say that a set Φ of formulas is *valid* in $\mathcal{F}$, in symbols $\mathcal{F} \models \Phi$, if all formulas of Φ are valid in $\mathcal{F}$. Now consider a class Σ of incidence frames. A formula ϕ is said to be *valid* in Σ, in symbols $\Sigma \models \phi$, if ϕ is valid in all incidence frames in Σ. We shall say that a set Φ of formulas is *valid* in Σ, in symbols $\Sigma \models \Phi$, if all formulas of Φ are valid in all incidence frames in Σ.

3 Definability and reductions

3.1 Definability

In this section we attack the problem of the definability of multifarious classes of incidence frames. A class Σ of incidence frames is said to be (*finitely*) *definable* within a class Δ of incidence frames if there exists a (finite) set Φ of formulas such that for all incidence frames $\mathcal{F}$ in Δ, $\mathcal{F}$ is in Σ iff $\mathcal{F} \models \Phi$.

PROPOSITION 6. *Let n be a natural integer such that $n \geq 2$. The class Σ^n of all n-rich incidence frames is finitely definable within the class Σ_{all} of all incidence frames. More precisely let*

- $RICH_0^n := [01](p_1, q_1) \wedge \ldots \wedge [01](p_{n-1}, q_{n-1}) \rightarrow [I]p_1 \vee \ldots \vee [I]p_{n-1} \vee \langle I\rangle(q_1 \wedge \ldots \wedge q_{n-1})$ *and*
- $RICH_1^n := [10](P_1, Q_1) \wedge \ldots \wedge [10](P_{n-1}, Q_{n-1}) \rightarrow [O]P_1 \vee \ldots \vee [O]P_{n-1} \vee \langle O\rangle(Q_1 \wedge \ldots \wedge Q_{n-1})$.

Then for all incidence frames $\mathcal{F}$, $\mathcal{F}$ is in Σ^n iff $\mathcal{F} \models \{\langle 01\rangle(\top, \top), \langle 10\rangle(\top, \top), RICH_0^n, RICH_1^n\}$.

The question lies open whether for all natural integers n such that $n \geq 2$, Σ^n is finitely definable within Σ_{all} in the language based on $[I]$ and $[O]$.

PROPOSITION 7. *The class Σ^ω of all ω-rich incidence frames is definable within Σ_{all}. More precisely for all incidence frames $\mathcal{F}$, $\mathcal{F}$ is in Σ^ω iff $\mathcal{F} \models \{\langle 01\rangle(\top, \top), \langle 10\rangle(\top, \top)\} \cup \{RICH_0^n\colon n \geq 2\} \cup \{RICH_1^n\colon n \geq 2\}$.*

The question lies open whether Σ^ω is definable within Σ_{all} in the language based on $[I]$ and $[O]$. Is Σ^ω finitely definable within Σ_{all}?

PROPOSITION 8. *The class Σ_c of all correct incidence frames is finitely definable within Σ^2. More precisely let*

- $C_0 := P \rightarrow [I]\langle O\rangle P$ *and*
- $C_1 := p \rightarrow [O]\langle I\rangle p$.

Then for all incidence frames $\mathcal{F}$ in Σ^2, $\mathcal{F}$ is in Σ_c iff $\mathcal{F} \models \{C_0, C_1\}$.

PROPOSITION 9. *The class Σ_p of all projective incidence frames is finitely definable within the class Σ_{rc}^2 of all rooted, 2-rich and correct incidence frames. More precisely let*

- $P_0 := [\cdot]P \rightarrow [\cdot][\cdot]P$ *and*
- $P_1 := [-]p \rightarrow [-][-]p$.

Then for all incidence frames $\mathcal{F}$ in Σ_{rc}^2, $\mathcal{F}$ is in Σ_p iff $\mathcal{F} \models \{P_0, P_1\}$.

PROPOSITION 10. *The class Σ_n of all normal incidence frames is finitely definable within the class Σ_{cp} of all correct and projective incidence frames. More precisely let*

- $N_0 := \odot P \rightarrow [\cdot](P \rightarrow [01]([O]P, [O]P))$ *and*
- $N_1 := \ominus p \rightarrow [-](p \rightarrow [10]([I]p, [I]p))$.

Then for all incidence frames $\mathcal{F}$ in Σ_{cp}, $\mathcal{F}$ is in Σ_n iff $\mathcal{F} \models \{N_0, N_1\}$.

The question lies open whether Σ_n is finitely definable within Σ_{cp} in the language based on $[I]$ and $[O]$.

PROPOSITION 11. *The class Σ_{Ar} of all Arguesian incidence frames is finitely definable within the class Σ_{cp}^2 of all 2-rich, correct and projective incidence frames. More precisely there exists a point formula Ar_0 and a line formula Ar_1 such that for all incidence frames $\mathcal{F}$ in Σ_{cp}^2, $\mathcal{F}$ is in Σ_{Ar} iff $\mathcal{F} \models \{Ar_0, Ar_1\}$.*

PROPOSITION 12. *The class Σ_{Pa} of all Pappian incidence frames is finitely definable within Σ^2_{cp}. More precisely there exists a point formula Pa_0 and a line formula Pa_1 such that for all incidence frames $\mathcal{F}$ in Σ^2_{cp}, $\mathcal{F}$ is in Σ_{Pa} iff $\mathcal{F} \models \{Pa_0, Pa_1\}$.*

3.2 Reductions

In this section we attack the problem of the reductions between several sublanguages of the language based on $[01]$, $[\in]$, $[I]$, $[\cdot]$, $\odot$, $[10]$, $[\ni]$, $[O]$, $[-]$ and $\ominus$. It may be of interest to note that for all incidence frames $\mathcal{F}$,

- $\mathcal{F} \models [01](p,q) \leftrightarrow [I]p \vee [I]q \vee [\in](p \wedge q)$ and
- $\mathcal{F} \models [10](P,Q) \leftrightarrow [O]P \vee [O]Q \vee [\ni](P \wedge Q)$.

Hence $[01]$ and $[10]$ can be defined as abbreviations in the language based on $[\in]$, $[I]$, $[\ni]$ and $[O]$ within Σ_{all}. Nevertheless

PROPOSITION 13. $[01]$ *and* $[10]$ *cannot be defined as abbreviations in the language based on $[I]$, $[\cdot]$, $\odot$, $[O]$, $[-]$ and $\ominus$ on the real projective plane $I\!R(2)$.*

PROPOSITION 14. $[01]$ *and* $[10]$ *cannot be defined as abbreviations in the language based on $[\in]$, $[\cdot]$, $\odot$, $[\ni]$, $[-]$ and $\ominus$ on $I\!R(2)$.*

It may be of interest to note that for all incidence frames $\mathcal{F}$,

- $\mathcal{F} \models [\cdot]P \leftrightarrow [I][O]P$ and
- $\mathcal{F} \models [-]p \leftrightarrow [O][I]p$.

Hence $[\cdot]$ and $[-]$ can be defined as abbreviations in the language based on $[I]$ and $[O]$ within Σ_{all}. Nevertheless

PROPOSITION 15.

1. *$[I]$ and $[O]$ cannot be defined as abbreviations in the language based on $[\in]$ and $[\ni]$ on $I\!R(2)$.*
2. *$[\cdot]$ and $[-]$ cannot be defined as abbreviations in the language based on $[\in]$ and $[\ni]$ on $I\!R(2)$.*
3. *$\odot$ and $\ominus$ cannot be defined as abbreviations in the language based on $[\in]$ and $[\ni]$ on $I\!R(2)$.*
4. *$[\in]$ and $[\ni]$ cannot be defined as abbreviations in the language based on $[I]$ and $[O]$ on $I\!R(2)$.*
5. *$\odot$ and $\ominus$ cannot be defined as abbreviations in the language based on $[I]$ and $[O]$ within the class Σ^{ω}_{cpn} of all ω-rich, correct, projective and normal incidence frames.*
6. *$[\in]$ and $[\ni]$ cannot be defined as abbreviations in the language based on $[\cdot]$ and $[-]$ on $I\!R(2)$.*

7. *$[I]$ and $[O]$ cannot be defined as abbreviations in the language based on $[\cdot]$ and $[-]$ on $\mathbb{R}(2)$.*

8. *$\odot$ and $\ominus$ cannot be defined as abbreviations in the language based on $[\cdot]$ and $[-]$ on $\mathbb{R}(2)$.*

9. *$[\in]$ and $[\ni]$ cannot be defined as abbreviations in the language based on $\odot$ and $\ominus$ on $\mathbb{R}(2)$.*

10. *$[I]$ and $[O]$ cannot be defined as abbreviations in the language based on $\odot$ and $\ominus$ on $\mathbb{R}(2)$.*

11. *$[\cdot]$ and $[-]$ cannot be defined as abbreviations in the language based on $\odot$ and $\ominus$ on $\mathbb{R}(2)$.*

The question lies open whether $\odot$ and $\ominus$ can be defined as abbreviations in the language based on $[I]$ and $[O]$ on $\mathbb{R}(2)$.

4 Axiomatization and completeness

4.1 Axiomatization

The set of all formulas that are valid in a class Σ of incidence frames, in symbols $L(\Sigma)$, is called the *logic* of Σ. In this section we attack the problem of the axiomatization of $L(\Sigma_{cpn}^{\omega})$. An *inference rule* (Φ, ϕ) consists of a set Φ of formulas and a formula ϕ. (Φ, ϕ) is said to be *finitary* if Φ is finite. An *inference system* (Φ, Ψ) consists of a recursive set Φ of formulas and a finite set Ψ of inference rules. We shall say that (Φ, Ψ) is *finite* if Φ is finite. (Φ, Ψ) is said to be *finitary* if all inference rules in Ψ are finitary. Let

- T_0 be the set of all point tautologies,
- T_1 be the set of all line tautologies,
- $K_0 := [I](p \to q) \to ([I]p \to [I]q)$ and
- $K_1 := [O](P \to Q) \to ([O]P \to [O]Q)$.

Let

- $MP_0 := (\{A, A \to B\}, B)$,
- $MP_1 := (\{a, a \to b\}, b)$,
- $US_0 := (\{A\}, A')$ where A' is obtained from A by uniformly replacing point variables in A by arbitrary point formulas and line variables in A by arbitrary line formulas,
- $US_1 := (\{a\}, a')$ where a' is obtained from a by uniformly replacing point variables in a by arbitrary point formulas and line variables in a by arbitrary line formulas,
- $G_0 := (\{a\}, [I]a)$ and

- $G_1 := (\{A\}, [O]A)$.

We shall say that a formula ϕ is *derivable* in (Φ, Ψ), in symbols $(\Phi, \Psi) \vdash \phi$, if ϕ belongs to the least set Φ' of formulas such that $\Phi \cup T_0 \cup T_1 \cup \{K_0, K_1\} \subseteq \Phi'$ and Φ' is closed under the inference rules in $\Psi \cup \{MP_0, MP_1, US_0, US_1, G_0, G_1\}$. A class Σ of incidence frames is said to be (*finitely*) *axiomatizable* if there exists a (finite) inference system (Φ, Ψ) such that for all formulas ϕ, ϕ is in $L(\Sigma)$ iff $(\Phi, \Psi) \vdash \phi$. Our aim is to find the axiomatization of the logics of some natural classes of incidence frames. Let $\Phi^{\omega}_{cpn} := \{SEP_0, SEP_1, \langle 01\rangle(\top, \top), \langle 10\rangle(\top, \top)\} \cup \{RICH^n_0\colon\ n \geq 2\} \cup \{RICH^n_1\colon n \geq 2\} \cup \{C_0, C_1, P_0, P_1, N_0, N_1\}$ and $\Psi^{\omega}_{cpn} := \emptyset$. Of course $(\Phi^{\omega}_{cpn}, \Psi^{\omega}_{cpn})$ is *sound* with respect to Σ^{ω}_{cpn}, i.e. $\{\phi\colon (\Phi^{\omega}_{cpn}, \Psi^{\omega}_{cpn}) \vdash \phi\} \subseteq L(\Sigma^{\omega}_{cpn})$. Nevertheless the question lies open whether $(\Phi^{\omega}_{cpn}, \Psi^{\omega}_{cpn})$ is *complete* with respect to Σ^{ω}_{cpn}, i.e. $L(\Sigma^{\omega}_{cpn}) \subseteq \{\phi\colon (\Phi^{\omega}_{cpn}, \Psi^{\omega}_{cpn}) \vdash \phi\}$. In particular the proof of the truth lemma for $(\Phi^{\omega}_{cpn}, \Psi^{\omega}_{cpn})$ does not seem to be feasible. How shall we overcome this difficulty? Our solution is based on the following

PROPOSITION 16. *Let p be a line variable that occurs neither in a nor in b and P be a point variable that occurs neither in A nor in B. Then*

- $\Sigma_{all} \models \forall p([I](a \vee p) \vee [I](b \vee \neg p)) \rightarrow [01](a, b)$ *and*
- $\Sigma_{all} \models \forall P([O](A \vee P) \vee [O](B \vee \neg P)) \rightarrow [10](A, B)$.

In the language without propositional quantifiers, we can imitate the above implications by additional inference rules:

- $R_0 := (\{A \rightarrow [I](a \vee p) \vee [I](b \vee \neg p)\colon\ p \in LV\}, A \rightarrow [01](a, b))$ and
- $R_1 := (\{a \rightarrow [O](A \vee P) \vee [O](B \vee \neg P)\colon\ P \in PV\}, a \rightarrow [10](A, B))$.

Remark that R_0 and R_1 are similar to the inference rule of intersection for the propositional dynamic logic with intersection of programs studied by Balbiani and Vakarelov [5]. Let ${\Psi^{\omega}_{cpn}}^+ := \Psi^{\omega}_{cpn} \cup \{R_0, R_1\}$. Of course $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^+)$ is sound with respect to Σ^{ω}_{cpn}. The main effect of R_0 and R_1 is to make possible the proof of the truth lemma for $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^+)$. Nevertheless the question lies open whether $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^+)$ is complete with respect to Σ^{ω}_{cpn}. In particular the proof that the rooted subframes of the canonical frame for $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^+)$ are in Σ^{ω}_{cpn} does not seem to be feasible. How shall we overcome this difficulty? Our solution is based on the following

PROPOSITION 17. *Let P be a point variable that does not occur in A and p be a line variable that does not occur in a. Then*

- $\Sigma_{all} \models \forall P(\odot P \rightarrow A) \rightarrow A$ *and*
- $\Sigma_{all} \models \forall p(\ominus p \rightarrow a) \rightarrow a$.

In the language without propositional quantifiers, we can imitate the above implications by additional inference rules:

- $D_0 := (\{\odot P \to A\colon\ P \in PV\}, A)$ and
- $D_1 := (\{\ominus p \to a\colon\ p \in LV\}, a)$.

Remark that D_0 and D_1 are similar to the inference rule of irreflexivity for the modal logic of inequality studied by de Rijke [14]. Let ${\Psi^{\omega}_{cpn}}^{++}$ be ${\Psi^{\omega}_{cpn}}^{+} \cup \{D_0, D_1\}$. Of course $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ is sound with respect to Σ^{ω}_{cpn}. The main effect of D_0 and D_1 is to make possible the proof that the rooted subframes of the canonical frame for $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ are in Σ^{ω}_{cpn}. Let us see how.

4.2 Completeness

The SD-theorem proved by Venema [17] implies the completeness of a wide variety of normal logics containing the inference rule of irreflexivity for the difference modality and satisfying some special conditions. Its proof uses techniques which are not necessary when we want to prove the completeness of a concrete normal logic. Moreover Venema's theorem has been obtained within the context of a one-sorted language. Hence it does not apply directly to our two-sorted language. In order to make this paper independent from Venema's paper we shall follow the line of reasoning suggested by Balbiani et al [2] and present our own proof of the completeness of $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ using ideas similar to Venema's ideas. In particular the proofs of lemmas 18–23 and theorem 24 are adaptations of the proofs of lemmas 3.5–3.11 in Balbiani et al [2]. Let $\flat$ and $\sharp$ be new symbols. Following the line of reasoning suggested by Goldblatt [11] we inductively define the *necessity forms* as follows:

- $\flat$ is a $(0,0)$-necessity form,
- $\sharp$ is a $(1,1)$-necessity form,
- if φ is a $(0,0)$-necessity form then $(A \to \varphi)$ is a $(0,0)$-necessity form and $[O]\varphi$ is a $(1,0)$-necessity form,
- if φ is a $(0,1)$-necessity form then $(A \to \varphi)$ is a $(0,1)$-necessity form and $[O]\varphi$ is a $(1,1)$-necessity form,
- if φ is a $(1,0)$-necessity form then $(a \to \varphi)$ is a $(1,0)$-necessity form and $[I]\varphi$ is a $(0,0)$-necessity form and
- if φ is a $(1,1)$-necessity form then $(a \to \varphi)$ is a $(1,1)$-necessity form and $[I]\varphi$ is a $(0,1)$-necessity form.

Note that each $(0,0)$-necessity form has a unique occurrence of $\flat$, each $(0,1)$-necessity form has a unique occurrence of $\sharp$, each $(1,0)$-necessity form has a unique occurrence of $\flat$ and each $(1,1)$-necessity form has a unique occurrence of $\sharp$. Now we introduce the following versions of R_0, R_1, D_0 and D_1:

- $R_{00} := (\{\varphi(A \to [I](a \vee p) \vee [I](b \vee \neg p))\colon\ p \in LV\}, \varphi(A \to [01](a, b)))$ where φ is a $(0,0)$-necessity form,

- $R_{01} := (\{\varphi(a \rightarrow [O](A \vee P) \vee [O](B \vee \neg P))\colon\ P \in PV\}, \varphi(a \rightarrow [10](A, B)))$ where φ is a $(0,1)$-necessity form,
- $R_{10} := (\{\varphi(A \rightarrow [I](a \vee p) \vee [I](b \vee \neg p))\colon\ p \in LV\}, \varphi(A \rightarrow [01](a, b)))$ where φ is a $(1,0)$-necessity form,
- $R_{11} := (\{\varphi(a \rightarrow [O](A \vee P) \vee [O](B \vee \neg P))\colon\ P \in PV\}, \varphi(a \rightarrow [10](A, B)))$ where φ is a $(1,1)$-necessity form,
- $D_{00} := (\{\varphi(\odot P \rightarrow A)\colon\ P \in PV\}, \varphi(A))$ where φ is a $(0,0)$-necessity form,
- $D_{01} := (\{\varphi(\ominus p \rightarrow a)\colon\ p \in LV\}, \varphi(a))$ where φ is a $(0,1)$-necessity form,
- $D_{10} := (\{\varphi(\odot P \rightarrow A)\colon\ P \in PV\}, \varphi(A))$ where φ is a $(1,0)$-necessity form and
- $D_{11} := (\{\varphi(\ominus p \rightarrow a)\colon\ p \in LV\}, \varphi(a))$ where φ is a $(1,1)$-necessity form.

LEMMA 18. *R_{00}, R_{01}, R_{10}, R_{11}, D_{00}, D_{01}, D_{10} and D_{11} are admissible in $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$, i.e. $\{\phi\colon (\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \vdash \phi\}$ is closed under R_{00}, R_{01}, R_{10}, R_{11}, D_{00}, D_{01}, D_{10} and D_{11}.*

A set X of point formulas will be called a *point theory* if $\{A\colon (\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \vdash A\} \subseteq X$ and X is closed under MP_0, R_{00}, R_{01}, D_{00} and D_{01} and a set x of line formulas will be called a *line theory* if $\{a\colon (\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \vdash a\} \subseteq x$ and x is closed under MP_1, R_{10}, R_{11}, D_{10} and D_{11}. Now consider a point theory X and a line theory x. X is said to be *consistent* if $\perp \notin X$ and x is said to be *consistent* if $\perp \notin x$. We shall say that X is *maximal* if it is consistent and for all point formulas A, $A \in X$ or $\neg A \in X$ and x is *maximal* if it is consistent and for all line formulas a, $a \in x$ or $\neg a \in x$. It can be proved that each maximal theory is a maximal consistent set of formulas in the classical sense and each maximal consistent set of formulas which is closed under the inference rules considered in lemma 18 is a maximal theory. Given a set X of point formulas and a set x of line formulas, let $X + A := \{B\colon A \rightarrow B \in X\}$, $x + a := \{b\colon a \rightarrow b \in x\}$, $[I]X := \{a\colon [I]a \in X\}$ and $[O]x := \{A\colon [O]A \in x\}$.

LEMMA 19. *Let X be a point theory and x be a line theory. Then*

- *$X + A$ is a point theory,*
- *$x + a$ is a line theory,*
- *$[I]X$ is a line theory and*
- *$[O]x$ is a point theory.*

Moreover $X + A$ is consistent iff $\neg A \notin X$ and $x + a$ is consistent iff $\neg a \notin x$.

LEMMA 20. *Let X be a point theory and x be a line theory. Then*

- *If X is consistent then it can be extended to a maximal point theory and*
- *if x is consistent then it can be extended to a maximal line theory.*

LEMMA 21. *Let X be a point theory and x be a line theory. Then*

- *If $[01](a,b) \notin X$ then there exists maximal line theories y, z such that $[I]X \subseteq y$, $[I]X \subseteq z$, $y \neq z$, $a \notin y$ and $b \notin z$ and*
- *if $[10](A,B) \notin x$ then there exists maximal point theories Y, Z such that $[O]x \subseteq Y$, $[O]x \subseteq Z$, $Y \neq Z$, $A \notin Y$ and $B \notin Z$.*

The *canonical frame* for $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ is the relational structure $\mathcal{F}_c = (Po_c, Li_c, I_c, O_c)$ where Po_c is the set of all maximal point theories, Li_c is the set of all maximal line theories, I_c is the binary relation between maximal point theories and maximal line theories defined by $I_c(X,x)$ iff $[I]X \subseteq x$ and O_c is the binary relation between maximal line theories and maximal point theories defined by $O_c(x,X)$ iff $[O]x \subseteq X$. Let π_c be the function assigning to each point variable $P \in PV$ the subset $\{X\colon P \in X\}$ of Po_c and λ_c be the function assigning to each line variable $p \in LV$ the subset $\{x\colon p \in x\}$ of Li_c.

LEMMA 22. *Let X be a maximal point theory and x be a maximal line theory. Then*

- *$(\mathcal{F}_c, \pi_c, \lambda_c), X \models A$ iff $A \in X$ and*
- *$(\mathcal{F}_c, \pi_c, \lambda_c), x \models a$ iff $a \in x$.*

LEMMA 23. *Let $\mathcal{F} = (Po, Li, I, O)$ be a rooted subframe of $\mathcal{F}_c$. Then $\mathcal{F}$ is in Σ^{ω}_{cpn}.*

THEOREM 24. *$(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ is sound and complete with respect to Σ^{ω}_{cpn}.*

Nevertheless

THEOREM 25. *$L(\Sigma^{\omega}_{cpn})$ cannot be finitely axiomatized with the inference rules in ${\Psi^{\omega}_{cpn}}^{++}$ in the language based on $[01]$ and $[10]$.*

To conclude this section we extend the results obtained so far to the problem of the axiomatization of the logic of multifarious classes of incidence frames. Let $\Phi^{\omega}_{cpnAr} := \Phi^{\omega}_{cpn} \cup \{Ar_0, Ar_1\}$ and $\Phi^{\omega}_{cpnPa} := \Phi^{\omega}_{cpn} \cup \{Pa_0, Pa_1\}$.

THEOREM 26.

- *$(\Phi^{\omega}_{cpnAr}, {\Psi^{\omega}_{cpn}}^{++})$ is sound and complete with respect to the class Σ^{ω}_{cpnAr} of all ω-rich, correct, projective, normal and Arguesian incidence frames and*
- *$(\Phi^{\omega}_{cpnPa}, {\Psi^{\omega}_{cpn}}^{++})$ is sound and complete with respect to the class Σ^{ω}_{cpnPa} of all ω-rich, correct, projective, normal and Pappian incidence frames.*

Nevertheless

THEOREM 27. *$L(\Sigma^{\omega}_{cpnAr})$ and $L(\Sigma^{\omega}_{cpnPa})$ cannot be finitely axiomatized with the inference rules in ${\Psi^{\omega}_{cpn}}^{++}$ in the language based on* [01] *and* [10].

Obviously $\mathbb{R}(2)$ is ω-rich, correct, projective, normal, Arguesian and Pappian. The question lies open whether $\mathbb{R}(2)$ is axiomatizable. For all natural integers n such that $n \geq 2$, let $\Phi^n_{cpn} = \{SEP_0, SEP_1, \langle 01\rangle(\top,\top), \langle 10\rangle(\top,\top), RICH^n_0, RICH^n_1, C_0, C_1, P_0, P_1, N_0, N_1\}$, $\Phi^n_{cpnAr} := \Phi^n_{cpn} \cup \{Ar_0, Ar_1\}$ and $\Phi^n_{cpnPa} := \Phi^n_{cpn} \cup \{Pa_0, Pa_1\}$.

THEOREM 28.

- $(\Phi^n_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ *is sound and complete with respect to the class Σ^n_{cpn} of all n-rich, correct, projective and normal incidence frames,*
- $(\Phi^n_{cpnAr}, {\Psi^{\omega}_{cpn}}^{++})$ *is sound and complete with respect to the class Σ^n_{cpnAr} of all n-rich, correct, projective, normal and Arguesian incidence frames and*
- $(\Phi^n_{cpnPa}, {\Psi^{\omega}_{cpn}}^{++})$ *is sound and complete with respect to the class Σ^n_{cpnPa} of all n-rich, correct, projective, normal and Pappian incidence frames.*

5 Conclusion

Our language can express multifarious modalities. The question lies open whether the modalities $\Box(A,B)$ "in all points X and Y collinear with and different from the current point, if X and Y are different then A is realized at X or B is realized at Y" and $\Box(a,b)$ "in all lines x and y coincident with and different from the current line, if x and y are different then a is realized at x or b is realized at y" can be expressed in our language.

Using the inference rules R_0, R_1, D_0 and D_1 we have given sound and complete axiomatizations of $L(\Sigma^{\omega}_{cpn})$, $L(\Sigma^{\omega}_{cpnAr})$ and $L(\Sigma^{\omega}_{cpnPa})$ and for all natural integers n such that $n \geq 2$, we have given sound and complete axiomatizations of $L(\Sigma^n_{cpn})$, $L(\Sigma^n_{cpnAr})$ and $L(\Sigma^n_{cpnPa})$. The question lies open whether the inference rules R_0, R_1, D_0 and D_1 can be eliminated in these axiomatizations and replaced with finitely many formulas.

Let us remark that according to a result of Venema [18] $NEXPTIME$ is a lower bound for the complexity of the satisfiability problems in Σ^{ω}_{cpn}, Σ^{ω}_{cpnAr} and Σ^{ω}_{cpnPa} and for all natural integers n such that $n \geq 2$, $NEXPTIME$ is a lower bound for the complexity of the satisfiability problems in Σ^n_{cpn}, Σ^n_{cpnAr} and Σ^n_{cpnPa}. The question lies open whether these satisfiability problems are decidable.

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Annex

Proof of lemmas 1–5 and propositions 6–12. Left to the reader. ■

Proof of proposition 13. Suppose there exists a point formula $A(p,q)$ in the language based on $[I]$, $[\cdot]$, $\odot$, $[O]$, $[-]$ and $\ominus$ such that $\mathbb{R}(2) \models [01](p,q) \leftrightarrow A(p,q)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\lambda(p) = Li \setminus I(X)$ and $\lambda(q) = Li \setminus I(X)$. Let y be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that y is not on X and $\lambda'(p) = Li \setminus (I(X) \cup \{y\})$ and $\lambda'(q) = Li \setminus (I(X) \cup \{y\})$. Let Z be a point in $\mathbb{R}(2)$ such that Z is in y. Remark that $\mathcal{M}, Z \models [01](p,q)$ and $\mathcal{M}', Z \not\models [01](p,q)$. We omit the easy proof that for all line formulas a in $\{p, \neg p, q, \neg q\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[I]a$, $[-]a$ and $\ominus a$. Hence $\mathcal{M}, Z \models A(p,q)$ iff $\mathcal{M}', Z \models A(p,q)$. Consequently $\mathcal{M}, Z \models [01](p,q)$ iff $\mathcal{M}', Z \models [01](p,q)$: a contradiction. ■

Proof of proposition 14. Suppose there exists a point formula $A(p,q)$ in the language based on $[\in]$, $[\cdot]$, $\odot$, $[\ni]$, $[-]$ and $\ominus$ such that $\mathbb{R}(2) \models [01](p,q) \leftrightarrow A(p,q)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\lambda(p) = \emptyset$ and $\lambda(q) = I(X)$. Let y be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that y is on X and $\lambda'(p) = \emptyset$ and $\lambda'(q) = I(X) \setminus \{y\}$. Remark that $\mathcal{M}, X \models [01](p,q)$ and $\mathcal{M}', X \not\models [01](p,q)$. It is readily checked that for all line formulas a in $\{p, \neg p, q, \neg q\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[\in]a$, $[-]a$ and $\ominus a$. Hence $\mathcal{M}, X \models A(p,q)$ iff $\mathcal{M}', X \models A(p,q)$. Consequently $\mathcal{M}, X \models [01](p,q)$ iff $\mathcal{M}', X \models [01](p,q)$: a contradiction. ■

Proof of proposition 15. 1. By proposition 14.
2. Suppose there exists a point formula $A(P)$ in the language based on $[\in]$ and $[\ni]$ such that $\mathbb{R}(2) \models [\cdot]P \leftrightarrow A(P)$. Let $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi(P) = Po$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi'(P) = Po \setminus \{X\}$. Remark that $\mathcal{M}, X \models [\cdot]P$ and $\mathcal{M}', X \not\models [\cdot]P$. We omit the easy proof that for all point formulas A in $\{P, \neg P\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[\ni]A$. Hence $\mathcal{M}, X \models A(P)$ iff $\mathcal{M}', X \models A(P)$. Consequently $\mathcal{M}, X \models [\cdot]P$ iff $\mathcal{M}', X \models [\cdot]P$: a contradiction.
3. Suppose there exists a point formula $A(P)$ in the language based on $[\in]$ and $[\ni]$ such that $\mathbb{R}(2) \models \odot P \leftrightarrow A(P)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi(P) = Po \setminus \{X\}$. Let $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi'(P) = Po$. Remark that $\mathcal{M}, X \models \odot P$ and $\mathcal{M}', X \not\models \odot P$. It is readily checked that for all point formulas A in $\{P, \neg P\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[\ni]A$. Hence $\mathcal{M}, X \models A(P)$ iff $\mathcal{M}', X \models A(P)$. Consequently $\mathcal{M}, X \models \odot P$ iff $\mathcal{M}', X \models \odot P$: a contradiction.
4. By proposition 13.
5. Suppose there exists a point formula $A(P)$ in the language based on $[I]$ and $[O]$ such that $\Sigma^{\omega}_{cpn} \models \odot P \leftrightarrow A(P)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi(P) = Po \setminus \{X\}$. Let $\mathcal{M}_1 = (Po_1, Li_1, I_1, O_1, \pi_1, \lambda_1)$ and $\mathcal{M}_2 = (Po_2, Li_2, I_2, O_2, \pi_2, \lambda_2)$ be disjoint copies of $\mathcal{M}$ and $\mathcal{M}' = (Po', Li', I', O', \pi', \lambda')$ be their disjoint unions. It is easy to see that $\mathcal{M}$ is a bounded morphic image of $\mathcal{M}'$ in the language based on $[I]$ and $[O]$. Obviously $\mathcal{M}'$ is ω-rich, correct and normal. Hence following the line of reasoning suggested by Venema [18] the reader may easily verify that $\mathcal{M}$ is a bounded morphic image of an ω-rich, correct, projective and normal incidence model $\mathcal{M}''$ containing $\mathcal{M}'$. Let X'' be a point in $\mathcal{M}''$ corresponding to X. Remark that $\mathcal{M}, X \models \odot P$ and $\mathcal{M}'', X'' \not\models \odot P$. Moreover for all point formulas A in the language based on $[I]$ and $[O]$, $\mathcal{M}, X \models A$ iff $\mathcal{M}'', X'' \models A$. Hence $\mathcal{M}, X \models A(P)$ iff $\mathcal{M}'', X'' \models A(P)$. Consequently $\mathcal{M}, X \models \odot P$ iff $\mathcal{M}'', X'' \models \odot P$: a contradiction.
6. Suppose there exists a point formula $A(p)$ in the language based on $[\cdot]$ and $[-]$ such that $\mathbb{R}(2) \models [\in]p \leftrightarrow A(p)$. Let x be a line in $\mathbb{R}(2)$

and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\lambda(p) = Li \setminus \{x\}$. Let y be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that $x \neq y$ and $\lambda'(p) = Li \setminus \{x, y\}$. Let Z be the unique point in $\mathbb{R}(2)$ such that Z is in x and Z is in y. Remark that $\mathcal{M}, Z \models [\in]p$ and $\mathcal{M}', Z \not\models [\in]p$. We omit the easy proof that for all line formulas a in $\{p, \neg p\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[-]a$. Hence $\mathcal{M}, Z \models A(p)$ iff $\mathcal{M}', Z \models A(p)$. Consequently $\mathcal{M}, Z \models [\in]p$ iff $\mathcal{M}', Z \models [\in]p$: a contradiction.

7. Suppose there exists a point formula $A(p)$ in the language based on $[\cdot]$ and $[-]$ such that $\mathbb{R}(2) \models [I]p \leftrightarrow A(p)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\lambda(p) = I(X)$. Let x be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that x is on X and $\lambda'(p) = I(X) \setminus \{x\}$. Remark that $\mathcal{M}, X \models [I]p$ and $\mathcal{M}', X \not\models [I]p$. It is readily checked that for all line formulas a in $\{p, \neg p\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[-]a$. Hence $\mathcal{M}, X \models A(p)$ iff $\mathcal{M}', X \models A(p)$. Consequently $\mathcal{M}, X \models [I]p$ iff $\mathcal{M}', X \models [I]p$: a contradiction.

8. Suppose there exists a point formula $A(P)$ in the language based on $[\cdot]$ and $[-]$ such that $\mathbb{R}(2) \models \odot P \leftrightarrow A(P)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\pi(P) = Po \setminus \{X\}$. Let Y be a point in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that $X \neq Y$ and $\pi'(P) = Po \setminus \{X, Y\}$. Remark that $\mathcal{M}, X \models \odot P$ and $\mathcal{M}', X \not\models \odot P$. It is easy to see that for all point formulas A in $\{P, \neg P\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $[\cdot]A$. Hence $\mathcal{M}, X \models A(P)$ iff $\mathcal{M}', X \models A(P)$. Consequently $\mathcal{M}, X \models \odot P$ iff $\mathcal{M}', X \models \odot P$: a contradiction.

9. Suppose there exists a point formula $A(p)$ in the language based on $\odot$ and $\ominus$ such that $\mathbb{R}(2) \models [\in]p \leftrightarrow A(p)$. Let X be a point in $\mathbb{R}(2)$, y be a line in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that X is in y and $\lambda(p) = I(X) \setminus \{y\}$. Let z be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that z is on X, $y \neq z$ and $\lambda'(p) = I(X) \setminus \{y, z\}$. Remark that $\mathcal{M}, X \models [\in]p$ and $\mathcal{M}', X \not\models [\in]p$. It is easy to see that for all line formulas a in $\{p, \neg p\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $\ominus a$. Hence $\mathcal{M}, X \models A(p)$ iff $\mathcal{M}', X \models A(p)$. Consequently $\mathcal{M}, X \models [\in]p$ iff $\mathcal{M}', X \models [\in]p$: a contradiction.

10. Suppose there exists a point formula $A(p)$ in the language based on $\odot$ and $\ominus$ such that $\mathbb{R}(2) \models [I]p \leftrightarrow A(p)$. Let X be a point in $\mathbb{R}(2)$ and $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an incidence model based on $\mathbb{R}(2)$ such that $\lambda(p) = I(X)$. Let y be a line in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that y is on X and $\lambda'(p) = I(X) \setminus \{y\}$. Remark that $\mathcal{M}, X \models [I]p$ and $\mathcal{M}', X \not\models [I]p$. It is easy to see that for all line formulas a in $\{p, \neg p\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $\ominus a$. Hence $\mathcal{M}, X \models A(p)$ iff $\mathcal{M}', X \models A(p)$. Consequently $\mathcal{M}, X \models [I]p$ iff $\mathcal{M}', X \models [I]p$: a contradiction.

11. Suppose there exists a point formula $A(P)$ in the language based on $\odot$ and $\ominus$ such that $\mathbb{R}(2) \models [\cdot]P \leftrightarrow A(P)$. Let $\mathcal{M} = (\mathbb{R}(2), \pi, \lambda)$ be an

incidence model based on $\mathbb{R}(2)$ such that $\pi(P) = Po$. Let X, Y be points in $\mathbb{R}(2)$ and $\mathcal{M}' = (\mathbb{R}(2), \pi', \lambda')$ be an incidence model based on $\mathbb{R}(2)$ such that $X \neq Y$ and $\pi'(P) = Po \setminus \{X, Y\}$. Remark that $\mathcal{M}, X \models [\cdot]P$ and $\mathcal{M}', X \not\models [\cdot]P$. It is easy to see that for all point formulas A in $\{P, \neg P\}$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $\odot A$. Hence $\mathcal{M}, X \models A(P)$ iff $\mathcal{M}', X \models A(P)$. Consequently $\mathcal{M}, X \models [\cdot]P$ iff $\mathcal{M}', X \models [\cdot]P$: a contradiction. ∎

Proof of proposition 16. Suppose $\Sigma_{all} \not\models \forall p([I](a \vee p) \vee [I](b \vee \neg p)) \rightarrow [01](a, b)$. Hence there exists an incidence frame $\mathcal{F} = (Po, Li, I, O)$, an incidence model $\mathcal{M} = (\mathcal{F}, \pi, \lambda)$ based on $\mathcal{F}$ and a point X such that $\mathcal{M}, X \models \forall p([I](a \vee p) \vee [I](b \vee \neg p))$ and $\mathcal{M}, X \not\models [01](a, b)$. Consequently there exists lines y, z such that $In(X, y, z)$, $\mathcal{M}, y \not\models a$ and $\mathcal{M}, z \not\models b$. Let λ' be the function assigning to each line variable $q \in LV$ different from p the subset $\lambda(q)$ of Li and assigning to p the subset $\{z\}$ of Li. Thus $(\mathcal{F}, \pi, \lambda'), X \models [I](a \vee p) \vee [I](b \vee \neg p)$. If $(\mathcal{F}, \pi, \lambda'), X \models [I](a \vee p)$ then $\mathcal{M}, y \models a$: a contradiction. If $(\mathcal{F}, \pi, \lambda'), X \models [I](b \vee \neg p)$ then $\mathcal{M}, z \models b$: a contradiction. ∎

Proof of proposition 17. Suppose $\Sigma_{all} \not\models \forall P(\odot P \rightarrow A) \rightarrow A$. Hence there exists an incidence frame $\mathcal{F} = (Po, Li, I, O)$, an incidence model $\mathcal{M} = (\mathcal{F}, \pi, \lambda)$ based on $\mathcal{F}$ and a point X such that $\mathcal{M}, X \models \forall P(\odot P \rightarrow A)$ and $\mathcal{M}, X \not\models A$. Let π' be the function assigning to each point variable $Q \in PV$ different from P the subset $\pi(Q)$ of Po and assigning to P the subset $Po \setminus \{X\}$ of Po. Consequently $(\mathcal{F}, \pi', \lambda), X \models \odot P \rightarrow A$. Thus $\mathcal{M}, X \models A$: a contradiction. ∎

Proof of lemma 18. The proof is based on the fact that Φ^{ω}_{cpn} contains C_0 and C_1. It can be done by induction on the length of necessity forms. ∎

Proof of lemma 19. The proof is based on the fact that point theories are closed under MP_0, R_{00}, R_{01}, D_{00} and D_{01} and line theories are closed under MP_1, R_{10}, R_{11}, D_{10} and D_{11} ∎

Proof of lemma 20. This is the Lindenbaum's lemma for $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$. Let X be a consistent point theory. Assuming that the set of all point formulas is arranged in some determinate order A_1, A_2, ..., we define the sequence X_0, X_1, ... of consistent point theories as follows. Let $X_0 = X$. Consider a positive integer n. If $X_{n-1} + A_n$ is consistent then we consider two cases. In the first case A_n cannot be represented as the negation of a conclusion of R_{00}, R_{01}, D_{00} and D_{01}. Hence let $X_n = X_{n-1} + A_n$. In the second case A_n can be represented as the negation of finitely many conclusions $\varphi_1(\phi_1)$, ..., $\varphi_k(\phi_k)$ of R_{00}, R_{01}, D_{00} and D_{01}. We define the sequence X^0_n, ..., X^k_n of consistent point theories as follows. Let $X^0_n = X_{n-1}$. Consider a positive integer i such that $i \leq k$. If $\varphi_i(\phi_i)$ is a conclusion of R_{00} then ϕ_i is a point formula of the form $A \rightarrow [01](a, b)$. Let $p \in LV$ be a line variable such that $X^{i-1}_n + \neg\varphi_i(A \rightarrow [I](a \vee p) \vee [I](b \vee \neg p))$ is consistent. Consequently let $X^i_n = X^{i-1}_n + \neg\varphi_i(A \rightarrow [I](a \vee p) \vee [I](b \vee \neg p))$. If $\varphi_i(\phi_i)$ is a conclusion of R_{01} then ϕ_i is a line formula of the form $a \to [10](A, B)$. Let $P \in PV$ be a point variable such that $X_n^{i-1} + \neg\varphi_i(a \to [O](A \vee P) \vee [O](B \vee \neg P))$ is consistent. Thus let $X_n^i = X_n^{i-1} + \neg\varphi_i(a \to [O](A \vee P) \vee [O](B \vee \neg P))$. If $\varphi_i(\phi_i)$ is a conclusion of D_{00} then ϕ_i is a point formula of the form A. Let $P \in PV$ be a point variable such that $X_n^{i-1} + \neg\varphi_i(\odot P \to A)$ is consistent. Therefore let $X_n^i = X_n^{i-1} + \neg\varphi_i(\odot P \to A)$. If $\varphi_i(\phi_i)$ is a conclusion of D_{01} then ϕ_i is a line formula of the form a. Let $p \in LV$ be a line variable such that $X_n^{i-1} + \neg\varphi_i(\ominus p \to a)$ is consistent. Hence let $X_n^i = X_n^{i-1} + \neg\varphi_i(\ominus p \to a)$. Proceed further in this way until k and let $X_n = X_n^k$. If $X_{n-1} + A_n$ is not consistent then let $X_n = X_{n-1}$. Then let $X' = X_0 \cup X_1 \cup \ldots$. The reader may easily verify that X' is a maximal point theory extending X. ∎

Proof of lemma 21. This is the diamond lemma for $(\Phi_{cpn}^{\omega}, {\Psi_{cpn}^{\omega}}^{++})$. Let X be a point theory. If $[01](a, b) \notin X$ then there exists a line variable $p \in LV$ such that $[I](a \vee p) \vee [I](b \vee \neg p) \notin X$. Hence $[I](a \vee p) \notin X$ and $[I](b \vee \neg p) \notin X$. Let $y = [I]X + \neg(a \vee p)$ and $z = [I]X + \neg(b \vee \neg p)$. Consequently y is a consistent line theory such that $[I]X \subseteq y$ and $\neg(a \vee p) \in y$ and z is a consistent line theory such that $[I]X \subseteq z$ and $\neg(b \vee \neg p) \in z$. By lemma 20 there exists maximal line theories y', z' such that $[I]X \subseteq y'$, $[I]X \subseteq z'$, $y' \neq z'$, $a \notin y'$ and $b \notin z'$. ∎

Proof of lemma 22. This is the truth lemma for $(\Phi_{cpn}^{\omega}, {\Psi_{cpn}^{\omega}}^{++})$. The proof can be done by induction on the length of formulas. The base case follows from the definition of π_c and λ_c. The Boolean cases follow from the classical properties of maximal consistent sets of formulas. It remains to deal with the modalities. The right to left direction is more or less immediate from the interpretation of [01] and the interpretation of [10]. If $(\mathcal{F}_c, \pi_c, \lambda_c), X \not\models [01](a, b)$ then there exists maximal line theories y, z such that $[I]X \subseteq y$, $[I]X \subseteq z$, $y \neq z$, $(\mathcal{F}_c, \pi_c, \lambda_c), y \not\models a$ and $(\mathcal{F}_c, \pi_c, \lambda_c), z \not\models b$. Hence there exists a line formula c such that $a \vee c \notin y$ and $b \vee \neg c \notin z$. Consequently $[I](a \vee c) \vee [I](b \vee \neg c) \notin X$. Since $[01](a, b) \to [I](a \vee c) \vee [I](b \vee \neg c) \in X$, then $[01](a, b) \notin X$. For the left to right direction suppose $[01](a, b) \notin X$. By lemma 21 there exists maximal line theories y, z such that $[I]X \subseteq y$, $[I]X \subseteq z$, $y \neq z$, $a \notin y$ and $b \notin z$. Thus $(\mathcal{F}_c, \pi_c, \lambda_c), y \not\models a$ and $(\mathcal{F}_c, \pi_c, \lambda_c), z \not\models b$. Therefore $(\mathcal{F}_c, \pi_c, \lambda_c), X \not\models [01](a, b)$. ∎

Proof of lemma 23. Firstly let us prove that $\mathcal{F}$ is 2-rich. If $\mathcal{F}$ is not 2-rich then there exists a point X such that $Card(I(X)) < 2$ or there exists a line x such that $Card(O(x)) < 2$. Without loss of generality let us assume that there exists a point X such that $Card(I(X)) < 2$. Since $\langle 01\rangle(\top, \top) \in X$, then by lemma 22 $(\mathcal{F}_c, \pi_c, \lambda_c), X \models \langle 01\rangle(\top, \top)$. Hence there exists lines y, z such that $In_c(X, y, z)$. Consequently $Card(I_c(X)) \geq 2$: a contradiction. Secondly let us prove that $\mathcal{F}$ is correct. If $\mathcal{F}$ is not correct then there exists a point X and a line x such that $I(X, x)$ and not $O(x, X)$ or there exists a line x and a point X such that $O(x, X)$ and not $I(X, x)$. Without loss of generality let us assume that there exists a point X and a line x such that

$I(X, x)$ and not $O(x, X)$. Hence $[I]X \subseteq x$ and $[O]x \not\subseteq X$. Consequently there exists a point formula A such that $[O]A \in x$ and $\neg A \in X$. Since $\neg A \to [I]\neg[O]A \in X$, then $[I]\neg[O]A \in X$. Thus $[O]A \notin x$: a contradiction. Thirdly let us prove that $\mathcal{F}$ is projective. If $\mathcal{F}$ is not projective then there exists points X, Y such that $I(X) \cap I(Y) = \emptyset$ or there exists lines x, y such that $O(x) \cap O(y) = \emptyset$. Without loss of generality let us assume that there exists points X, Y such that $I(X) \cap I(Y) = \emptyset$. Since $\mathcal{F}$ is 2-rich, then $X \neq Y$. Let $Q \in PV$ be a point variable such that $\odot Q \in Y$. Since $\mathcal{F}$ is rooted and correct, then there exists a natural integer n such that $Y \in (I \circ O)^n(X)$. Hence $\langle \cdot \rangle^n \odot Q \in X$. Since $\langle \cdot \rangle^n \odot Q \to \neg[01](\neg\langle O \rangle \odot Q, \bot) \in X$, then $[01](\neg\langle O \rangle \odot Q, \bot) \notin X$. By lemma 21 there exists a maximal line theory z such that $[I]X \subseteq z$ and $\langle O \rangle \odot Q \in z$. If $[I]Y \not\subseteq z$ then there exists a line formula a such that $[I]a \in Y$ and $a \notin z$. Consequently $\langle \cdot \rangle^n(\odot Q \wedge [I]a) \in X$. Since $\langle \cdot \rangle^n(\odot Q \wedge [I]a) \to [I][O](\odot Q \to [I]a) \in X$, then $[I][O](\odot Q \to [I]a) \in X$. Thus $[O](\odot Q \to [I]a) \in z$. Therefore $\langle O \rangle [I]a \in z$. Since $\langle O \rangle [I]a \to a \in z$, then $a \in z$: a contradiction. Hence $I(X) \cap I(Y) \neq \emptyset$: a contradiction.

Fourthly let us prove that $\mathcal{F}$ is normal. If $\mathcal{F}$ is not normal then there exists points X, Y and lines z, t such that $I(X, z)$, $I(Y, z)$, $I(X, t)$, $I(Y, t)$, $X \neq Y$ and $z \neq t$ or there exists lines x, y and points Z, T such that $O(x, Z)$, $O(y, Z)$, $O(x, T)$, $O(y, T)$, $x \neq y$ and $Z \neq T$. Without loss of generality let us assume that there exists points X, Y and lines z, t such that $I(X, z)$, $I(Y, z)$, $I(X, t)$, $I(Y, t)$, $X \neq Y$ and $z \neq t$. Let $P \in PV$ be a point variable such that $\odot P \in X$. Hence $P \notin X$ and $[01]([10](P, P), \bot) \in X$. By lemma 22 $(\mathcal{F}_c, \pi_c, \lambda_c), X \not\models P$ and $(\mathcal{F}_c, \pi_c, \lambda_c), X \models [01]([10](P, P), \bot)$. Consequently $(\mathcal{F}_c, \pi_c, \lambda_c), z \models [10](P, P)$. Thus $(\mathcal{F}_c, \pi_c, \lambda_c), Y \models P$. By lemma 22 $P \in Y$. Since $\odot P \to [I][O](P \to [01]([O]P, [O]P)) \in X$, then $[I][O](P \to [01]([O]P, [O]P)) \in X$. Therefore $[O](P \to [01]([O]P, [O]P)) \in z$. Hence $P \to [01]([O]P, [O]P) \in Y$. Consequently $[01]([O]P, [O]P) \in Y$. By lemma 22 $(\mathcal{F}_c, \pi_c, \lambda_c), Y \models [01]([O]P, [O]P)$. Thus $(\mathcal{F}_c, \pi_c, \lambda_c), z \models [O]P$ or $(\mathcal{F}_c, \pi_c, \lambda_c), t \models [O]P$. If $(\mathcal{F}_c, \pi_c, \lambda_c), z \models [O]P$ then $(\mathcal{F}_c, \pi_c, \lambda_c), X \models P$: a contradiction. If $(\mathcal{F}_c, \pi_c, \lambda_c), t \models [O]P$ then $(\mathcal{F}_c, \pi_c, \lambda_c), X \models P$: a contradiction.

Fifthly let us prove that $\mathcal{F}$ is ω-rich. If $\mathcal{F}$ is not ω-rich then there exists a natural integer n such that $n \geq 2$ and $\mathcal{F}$ is not n-rich. Hence there exists a point X such that $Card(I(X)) < n$ or there exists a line x such that $Card(O(x)) < n$. Without loss of generality let us assume that there exists a point X such that $Card(I(X)) < n$. Consequently there exists lines y_1, …, y_{n-1} such that $I(X) = \{y_1, \ldots, y_{n-1}\}$. Since $\mathcal{F}$ is 2-rich, then $n-1 \geq 2$. Let $p_1 \in LV, \ldots, p_{n-1} \in LV$ be line variables such that $\ominus p_1 \in y_1$, …, $\ominus p_{n-1} \in y_{n-1}$. The reader may easily verify that $(\mathcal{F}_c, \pi_c, \lambda_c), X \models [01](p_1, p_1) \wedge \ldots \wedge [01](p_{n-1}, p_{n-1})$ and $(\mathcal{F}_c, \pi_c, \lambda_c), X \not\models [I]p_1 \vee \ldots \vee [I]p_{n-1} \vee \langle I \rangle(p_1 \wedge \ldots \wedge p_{n-1})$. By lemma 22 $[01](p_1, p_1) \wedge \ldots \wedge [01](p_{n-1}, p_{n-1}) \in X$. Since $[01](p_1, p_1) \wedge \ldots \wedge [01](p_{n-1}, p_{n-1}) \to [I]p_1 \vee \ldots \vee [I]p_{n-1} \vee \langle I \rangle(p_1 \wedge \ldots \wedge p_{n-1}) \in X$, then $[I]p_1 \vee \ldots \vee [I]p_{n-1} \vee \langle I \rangle(p_1 \wedge \ldots \wedge p_{n-1}) \in X$. By lemma 22 $(\mathcal{F}_c, \pi_c, \lambda_c), X \models [I]p_1 \vee \ldots \vee [I]p_{n-1} \vee \langle I \rangle(p_1 \wedge \ldots \wedge p_{n-1})$: a contradiction. ■

Proof of theorem 24. Let A be a point formula such that $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \not\vdash A$. By lemmas 19 and 20 there exists a maximal point theory X such that $A \notin X$. By lemma 22 $(\mathcal{F}_c, \pi_c, \lambda_c), X \not\models A$. Let $\mathcal{F}$ be the subframe of $\mathcal{F}_c$ rooted at X. Hence $\mathcal{F} \not\models A$. By lemma 23 $\mathcal{F}$ is in Σ^{ω}_{cpn}. Consequently $\Sigma^{\omega}_{cpn} \not\models A$. ■

Proof of theorem 25. If $L(\Sigma^{\omega}_{cpn})$ can be finitely axiomatized with the inference rules in ${\Psi^{\omega}_{cpn}}^{++}$ in the language based on [01] and [10] then there exists a point formula A and a line formula a such that for all formulas ϕ, ϕ is in $L(\Sigma^{\omega}_{cpn})$ iff $(\{A, a\}, {\Psi^{\omega}_{cpn}}^{++}) \vdash \phi$. Hence A is in $L(\Sigma^{\omega}_{cpn})$ and a is in $L(\Sigma^{\omega}_{cpn})$. Consequently $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \vdash A$ and $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++}) \vdash a$. Let n be the greatest natural integer such that $n \geq 2$ and $RICH^n_0$ or $RICH^n_1$ are necessarily used in the derivations of A or a in $(\Phi^{\omega}_{cpn}, {\Psi^{\omega}_{cpn}}^{++})$ and $\Phi^{n}_{cpn} = \{SEP_0, SEP_1, \langle 01\rangle(\top, \top), \langle 10\rangle(\top, \top), RICH^n_0, RICH^n_1, C_0, C_1, P_0, P_1, N_0, N_1\}$. Thus for all incidence frames $\mathcal{F}$, if $\mathcal{F} \models \Phi^n_{cpn}$ then $\mathcal{F} \models \Phi^{\omega}_{cpn}$. Let p be the least prime number such that $p \geq n$. It is a well-known fact that there exists a p-rich, correct, projective and normal incidence frame $\mathcal{F}$ such that $\mathcal{F}$ is not ω-rich. By lemma 1 and propositions 6, 8, 9 and 10 $\mathcal{F} \models \Phi^n_{cpn}$. Therefore $\mathcal{F} \models \Phi^{\omega}_{cpn}$. By proposition 7 $\mathcal{F} \not\models \Phi^{\omega}_{cpn}$: a contradiction. ■

Proof of theorem 26. Let $\mathcal{F}_{cAr}$ be the canonical frame for $(\Phi^{\omega}_{cpnAr}, {\Psi^{\omega}_{cpn}}^{++})$, $\mathcal{F}_{cPa}$ be the canonical frame for $(\Phi^{\omega}_{cpnPa}, {\Psi^{\omega}_{cpn}}^{++})$, $\mathcal{F}_{Ar}$ be a rooted subframe of $\mathcal{F}_{cAr}$ and $\mathcal{F}_{Pa}$ be a rooted subframe of $\mathcal{F}_{cPa}$. Obviously lemmas 18–22 still hold for $(\Phi^{\omega}_{cpnAr}, {\Psi^{\omega}_{cpn}}^{++})$ and $(\Phi^{\omega}_{cpnPa}, {\Psi^{\omega}_{cpn}}^{++})$. Hence following the line of reasoning suggested in the proof of lemma 23 the reader may easily verify that $\mathcal{F}_{Ar}$ is Arguesian and $\mathcal{F}_{Pa}$ is Pappian. Consequently following the line of reasoning suggested in the proof of theorem 24 the reader may easily verify that $(\Phi^{\omega}_{cpnAr}, {\Psi^{\omega}_{cpn}}^{++})$ is sound and complete with respect to Σ^{ω}_{cpnAr} and $(\Phi^{\omega}_{cpnPa}, {\Psi^{\omega}_{cpn}}^{++})$ is sound and complete with respect to Σ^{ω}_{cpnPa}. ■

Proof of theorem 27. The proof can be easily adapted from the proof of theorem 25. ■

Proof of theorem 28. The proof can be easily adapted from the proof of theorems 24 and 26. ■

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