A Kripke-Joyal Semantics for Noncommutative Logic in Quantales

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ABSTRACT. A structural semantics is developed for a first-order logic, with infinite disjunctions and conjunctions, that is characterised algebraically by quantales. The model structures involved combine the "covering systems" approach of Kripke-Joyal intuitionistic semantics from topos theory with the ordered groupoid structures used to model various connectives in substructural logics. The latter are used to interpret the noncommutative quantal conjunction & ("and then") and its residual implication connectives.

The completeness proof uses the MacNeille completion and the theory of quantic nuclei to first embed a residuated semigroup into a quantale, and then represent the quantale as an algebra of subsets of a model structure.

The final part of the paper makes some observations about quantal modal logic, giving in particular a structural modelling of the logic of closure operators on quantales.

Keywords: quantale, noncommutative conjunction, quantic nucleus, Kripke-Joyal semantics, infinitary proof theory, modality.

1 Introduction

A *locale* is a complete lattice in which finite meets distribute over arbitrary joins, the motivating example being the lattice of open subsets of a topological space. Any locale is a Heyting algebra – with the relative pseudocomplement $a \Rightarrow b$ being the join of $\{x : x \sqcap a \leq b\}$ – so it provides algebraic semantics for infinitary first-order intuitionistic logic, with $\Rightarrow$ interpreting implication, lattice joins and meets interpreting disjunctions $\bigvee$ and conjunctions $\bigwedge$ (possibly infinite), and the quantifiers $\exists$ and $\forall$ being treated as special disjunctions and conjunctions, respectively.

The notion of a *quantale* was introduced by C. J. Mulvey [13] to give a noncommutative extension of the locale concept that could be applied to spaces related to the foundations of quantum theory, such as the spectra of C*-algebras. A quantale is a complete lattice with an associative (but possibly not commutative) operation $a \bullet b$ that distributes over joins in each argument. A locale is then just a quantale in which $a \bullet b$ is the lattice meet of a and b. Mulvey suggested that $\bullet$ should interpret a logical connective & that is a kind of "sequential conjunction" with "a vestige of temporality in its interpretation". A propositional formula $\varphi \& \psi$ is to be read "φ *and*

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then ψ". See [14] for a representation of the spectrum of a C*-algebra as the Lindenbaum algebra of a propositional logic of formulas built using this & and $\bigvee$. The paper [16] gives a novel application of propositional logic in quantales to the classification of Penrose tilings of the plane. Further information about the way that quantales generalise locales is given in [15].

Since a quantale is complete, it can still interpret all of $\bigvee$, $\bigwedge$, $\exists$ and $\forall$. It no longer has a Heyting implication $\Rightarrow$ (unless $\bullet$ is commutative), but instead $\bullet$ has left and right *residual* operations, $\Rightarrow_l$ and $\Rightarrow_r$, which can be used to interpret two connectives, $\rightarrow_l$ and $\rightarrow_r$, which we think of as left and right implication. The aim of this paper is to develop a semantics for the logic of all these connectives as interpreted in quantales, by combining the idea of the Kripke-Joyal intuitionistic semantics for $\bigvee, \bigwedge, \exists, \forall$ arising from topos theory [11, 3] with the models for $\&, \rightarrow_l, \rightarrow_r$ in substructural logics that are based on ordered groupoids $\langle S, \leq, \cdot \rangle$ [19, 18, 6, 7].

It has long been recognised[1] that a binary connective like & can be modelled by a ternary relation R on Kripke-type models, with the semantics

$$x \models \varphi \& \psi \quad \text{iff} \quad \exists y \exists z : Rxyz \text{ and } y \models \varphi \text{ and } z \models \psi.$$

This relates naturally to the "and then" reading of & if we view $Rxyz$ as a relation of relativistic temporal ordering, meaning that y precedes z in time from the perspective of x. In the groupoid formalism, $Rxyz$ becomes the condition that $y \cdot z \leq x$. The Kripke-Joyal semantics uses collections of sets called "covers" in a way that it is formally similar to the neighbourhood semantics of modal logics. We define a notion of *model structure* as a preordered groupoid, a semigroup in fact, with a covering system obeying axioms that interact the covers with the ordering and the semigroup structure.

This infinitary first-order logic of quantales is axiomatised by constructing Lindenbaum algebras that are residuated semigroups and then embedding these in quantales by means of the MacNeille completion and the theory of quantic nuclei. We then show that any quantale can be represented as an algebra of subsets of a model structure, and read off the Kripke-Joyal semantics from this. The final section of the paper makes a foray into the world of quantal modal logic, giving in particular a structural modelling of the logic of closure operators on quantales.

2 Posemigroups and Quantales

Given a *poset* $\langle S, \leq \rangle$, comprising a partial ordering $\leq$ on a set S, we write $\sum X$ for the *join* (=least upper bound), and $\prod X$ for the *meet* (=greatest lower bound), of a set $X \subseteq S$, when these bounds exist. A poset is *complete* if every subset has a join, or equivalently if every subset has a meet.

A *posemigroup* $\mathcal{S} = \langle S, \leq, \bullet \rangle$ has an associative binary operation $\bullet$ that is monotone (i.e. order preserving) in each argument, meaning that $x \leq z$

[1] The idea goes back to Jónsson and Tarski. The ternary relation semantics is most associated with the Routley-Meyer semantics of the (commutative) fusion connective in relevant logic. See also [7].

implies $x \bullet y \leq z \bullet y$ and $y \bullet x \leq y \bullet z$. A *quantale* is a complete poset with an associative $\bullet$ in which the equations

$$(\sum X) \bullet a = \sum_{x \in X}(x \bullet a) \tag{2.1}$$

$$a \bullet (\sum X) = \sum_{x \in X}(a \bullet x) \tag{2.2}$$

hold for every set $X \subseteq S$. These equations imply that $\bullet$ is monotone.

A posemigroup $\mathcal{S}$ is *residuated* if there are binary operations $\Rightarrow_l$ and $\Rightarrow_r$ on S, called the *left and right residuals* of $\bullet$, satisfying

$$x \bullet a \leq b \quad \text{iff} \quad x \leq a \Rightarrow_l b \tag{2.3}$$

$$a \bullet x \leq b \quad \text{iff} \quad x \leq a \Rightarrow_r b. \tag{2.4}$$

These two residual operations are identical precisely when $\bullet$ is commutative. In a residuated semigroup, the equations (2.1) and (2.2) hold whenever the joins they refer to exist. This is a well-known fact, and is really an instance of the general categorical result that left adjoint functors preserve colimits: here the map $x \mapsto x \bullet a$ is a functor on the poset category $\langle S, \leq \rangle$ that is left adjoint to $b \mapsto a \Rightarrow_l b$ by (2.3), while $x \mapsto a \bullet x$ is left adjoint to $b \mapsto a \Rightarrow_r b$ by (2.4).

Thus a complete residuated posemigroup is a quantale. The converse is also true: every quantale is residuated with $a \Rightarrow_l b = \sum\{x : x \bullet a \leq b\}$ and $a \Rightarrow_r b = \sum\{x : a \bullet x \leq b\}$.

Any residuated posemigroup can be embedded into a quantale by the famous completion construction of MacNeille [12]. This is shown in Section 4 of [18], where it is inferred from a more abstract result, and in Chapter 8 of [21] in the commutative case. Here we give the details of the proof in a way that emphasises its dependence on both residual operations when $\bullet$ is not commutative. The construction uses the theory of closure operators: a *closure operator* on a poset S is a function $j : S \to S$ that is *monotone*: $x \leq y$ implies $jx \leq jy$; *inflationary*: $x \leq jx$; and *idempotent*: $jjx = jx$. An element x is j*-closed* if $jx = x$. If S is complete, then the set S^j of j-closed elements is closed under meets $\prod X$, and so is complete under the same partial ordering. The join operation $\sum^j$ in S^j is given by $\sum^j X = j(\sum X)$.

Now if X is a subset of a poset S, let lX be the set of all lower bounds, and uX the set of all upper bounds, of X in S. Put $mX = luX$. Then m is a closure operator on the complete poset $\langle \mathcal{P}S, \subseteq \rangle$, where $\mathcal{P}S$ is the powerset of S. Any set of the form lX is m-closed, including the set $x{\downarrow} = l\{x\} = \{y \in S : y \leq x\}$ for each $x \in S$. So the set $(\mathcal{P}S)^m$ of all m-closed subsets of S is complete under the ordering $\subseteq$, as explained above, with $\prod \mathcal{X} = \bigcap \mathcal{X}$ and $\sum \mathcal{X} = m(\bigcup \mathcal{X})$ in $(\mathcal{P}S)^m$, where $\mathcal{X}$ is any collection of m-closed sets. The function $f_m(x) = x{\downarrow}$ is an order-invariant injection $f_m : \langle S, \leq \rangle \to \langle (\mathcal{P}S)^m, \subseteq \rangle$ having the crucial property that it *preserves any joins and meets that exist in* S (see [5, pp. 40–44] for a comprehensive discussion of this MacNeille completion construction).

A *quantic nucleus* is a closure operator on a posemigroup that satisfies

$$jx \bullet jy \leq j(x \bullet y). \tag{2.5}$$

If j is a quantic nucleus on a quantale $\langle Q, \leq, \bullet \rangle$, then the complete poset $\langle Q^j, \leq \rangle$ of j-closed elements is a quantale under the operation $a \bullet_j b = j(a \bullet b)$ [17, Theorem 2.1]. Moreover Q^j is closed under the residuals of $\bullet$, and indeed both $a \Rightarrow_l b$ and $a \Rightarrow_r b$ belong to Q^j whenever $b \in Q^j$ [20, Prop. 3.1.2]. From this it can be shown that the residuals of $\bullet_j$ on Q^j are just the restrictions of the residuals of $\bullet$ on Q to Q^j.

Now from any semigroup $\langle S, \bullet \rangle$ we can construct a quantale $\langle \mathcal{P}S, \subseteq, \bullet \rangle$ on the powerset of S by putting

$$X \bullet Y = \{x \bullet y : x \in X \text{ and } y \in Y\}$$

for all $X, Y \subseteq S$. In this quantale the join $\sum \mathcal{X}$ is the set-theoretic union $\bigcup \mathcal{X}$, and the residuals are given by

$$X \Rightarrow_l Y = \{z \in S : \{z\} \bullet X \subseteq Y\}, \ X \Rightarrow_r Y = \{z \in S : X \bullet \{z\} \subseteq Y\}. \quad (2.6)$$

LEMMA 1. *For any residuated semigroup $\langle S, \leq, \bullet \rangle$, the MacNeille closure operator $mX = luX$ is a quantic nucleus on the quantale $\langle \mathcal{P}S, \subseteq, \bullet \rangle$, and so $\langle (\mathcal{P}S)^m, \subseteq, \bullet_m \rangle$ is a quantale. Moreover the injection $f_m : S \to (\mathcal{P}S)^m$ preserves $\bullet$ and its residuals.*

Proof. We have to show that $(luX) \bullet (luY) \subseteq lu(X \bullet Y)$, so fix any $x \in luX$ and $y \in luY$. Let $z \in u(X \bullet Y)$. We have to show $x \bullet y \leq z$.

Now if $y' \in Y$, then for all $x' \in X$, $x' \bullet y' \leq z$ and hence $x' \leq y' \Rightarrow_l z$. This shows that $y' \Rightarrow_l z \in uX$, so $x \leq y' \Rightarrow_l z$ as $x \in luX$, hence $x \bullet y' \leq z$ and so $y' \leq x \Rightarrow_r z$. Since that holds for all $y' \in Y$, $x \Rightarrow_r z \in uY$, so $y \leq x \Rightarrow_r z$, implying $x \bullet y \leq z$ as required. Thus (2.5) holds when $j = m$.

For preservation of $\bullet$ by f_m, note that since $x \bullet y \in (x\downarrow) \bullet (y\downarrow)$ we get $(x \bullet y)\downarrow \subseteq m((x\downarrow) \bullet (y\downarrow))$. But $(x\downarrow) \bullet (y\downarrow) \subseteq (x \bullet y)\downarrow$, so $m((x\downarrow) \bullet (y\downarrow)) \subseteq m((x \bullet y)\downarrow) = (x \bullet y)\downarrow$. Hence $(x \bullet y)\downarrow = m((x\downarrow) \bullet (y\downarrow)) = (x\downarrow) \bullet_m (y\downarrow)$ as required.

Now the residuals of $\bullet_m$ are just the restrictions of the residuals of $\bullet$, so these are given on $(\mathcal{P}S)^m$ by (2.6). Preservation of $\Rightarrow_l$ thus amounts to the condition that $z \leq x \Rightarrow_l y$ iff $z \bullet (x\downarrow) \subseteq (y\downarrow)$, which follows readily by (2.3) and monotonicity of $\bullet$. Preservation of $\Rightarrow_r$ follows similarly from (2.4). ■

COROLLARY 2. *Every residuated posemigroup has an isomorphic embedding into the residuated posemigroup of a quantale that preserves any existing joins and meets.*

3 Logic

We assume familiarity with the syntactic apparatus of first-order logic with infinite disjunctions and conjunctions. Sensitivity to the distinction between *large classes* and *sets* (small classes) is required, since collections of formulas may be large.

3.1 Formulas

Fix a denumerable list $v_0, \ldots, v_n, \ldots$ of individual variables and a set of predicate letters, with typical member P, that are k-ary for various $k < \omega$. These are used to define *atomic* formulas $P(v_{n_1}, \ldots, v_{n_k})$. A *preformula* is any expression generated from atomic formulas by using the binary connectives $\&, \to_l, \to_r$ and the quantifiers $\exists v_n$, $\forall v_n$, and by allowing the formation of the disjunction $\bigvee \Phi$ and conjunction $\bigwedge \Phi$ of any *set* Φ of formulas. A *formula* is a preformula that has only finitely many free variables. This constraint is a standard convention in infinitary logic, designed to avoid dealing with expressions that have too many free variables to be convertible into sentences by prefixing quantifiers [1, 4]. We confine our attention to formulas throughout.

The class of all formulas is large, so cannot be used in its entirety to build a Lindenbaum algebra as a quotient set (the requisite equivalence classes of formulas may themselves be large). The class $\mathrm{sub}\,\varphi$ of subformulas of a formula φ is defined in the usual way, e.g. if $\varphi = \bigvee \Phi$, then $\mathrm{sub}\,\varphi = \{\varphi\} \cup \bigcup_{\psi \in \Phi} \mathrm{sub}\,\psi$. Then if Ψ is any set of formulas, the class $\mathrm{sub}\,\Psi = \bigcup_{\psi \in \Psi} \mathrm{sub}\,\psi$ of all subformulas of members of Ψ is a set.

We adopt the usual conventions for variable-substitution, writing $\varphi(w/v)$ for the formula obtained by substituting w for all free occurrences of v in a suitable alphabetic variant of φ.

3.2 Quantal Models

These are structures $\mathfrak{A} = \langle \mathcal{Q}, D, V \rangle$ with $\mathcal{Q} = \langle Q, \leq, \bullet \rangle$ being a quantale, D a non-empty set of individuals, and V a function assigning to each k-ary predicate letter P a function $V(P) : D^k \to Q$. To interpret variables in the model we use D-*valuations*, which are sequences $\sigma = \langle \sigma_0, \ldots, \sigma_n, \ldots \rangle$ of elements of D, the idea being that σ assigns value σ_n to variable v_n. We write $\sigma(d/n)$ for the valuation obtained from σ by replacing σ_n by d. For each formula φ we specify a value $\|\varphi\|^{\mathfrak{A}}_\sigma \in Q$ for each valuation σ. This is defined inductively on the formation of φ, as follows:

- $\|P(v_{n_1}, \ldots, v_{n_k})\|^{\mathfrak{A}}_\sigma = V(P)(\sigma_{n_1}, \ldots, \sigma_{n_k})$
- $\|\varphi \& \psi\|^{\mathfrak{A}}_\sigma = \|\varphi\|^{\mathfrak{A}}_\sigma \bullet \|\psi\|^{\mathfrak{A}}_\sigma$
- $\|\varphi \to_l \psi\|^{\mathfrak{A}}_\sigma = \|\varphi\|^{\mathfrak{A}}_\sigma \Rightarrow_l \|\psi\|^{\mathfrak{A}}_\sigma, \qquad \|\varphi \to_r \psi\|^{\mathfrak{A}}_\sigma = \|\varphi\|^{\mathfrak{A}}_\sigma \Rightarrow_r \|\psi\|^{\mathfrak{A}}_\sigma$
- $\|\bigvee \Phi\|^{\mathfrak{A}}_\sigma = \sum_{\varphi \in \Phi} \|\varphi\|^{\mathfrak{A}}_\sigma$
- $\|\bigwedge \Phi\|^{\mathfrak{A}}_\sigma = \prod_{\varphi \in \Phi} \|\varphi\|^{\mathfrak{A}}_\sigma$
- $\|\exists v_n \varphi\|^{\mathfrak{A}}_\sigma = \sum_{d \in D} \|\varphi\|^{\mathfrak{A}}_{\sigma(d/n)}$
- $\|\forall v_n \varphi\|^{\mathfrak{A}}_\sigma = \prod_{d \in D} \|\varphi\|^{\mathfrak{A}}_{\sigma(d/n)}$.

We write $\varphi \models^{\mathfrak{A}} \psi$ if $\|\varphi\|^{\mathfrak{A}}_\sigma \leq \|\psi\|^{\mathfrak{A}}_\sigma$ for all D-valuations σ, and say that φ *semantically implies* ψ *over quantales*, written $\varphi \models^q \psi$, if $\varphi \models^{\mathfrak{A}} \psi$ for all quantal models $\mathfrak{A}$.

3.3 Proof Theory

A *sequent* is an expression $\varphi \vdash \psi$ with φ and ψ being formulas. Alternatively, a sequent may be thought of as an ordered pair of formulas, with the symbol $\vdash$ denoting a class of sequents, i.e. a binary relation between formulas.

Let $\vdash_q$ be the smallest class of sequents that includes all instances of the axiom schemas

- $\varphi \vdash \varphi$;
- $\varphi\&(\psi\&\rho) \vdash (\varphi\&\psi)\&\rho, \qquad (\varphi\&\psi)\&\rho \vdash \varphi\&(\psi\&\rho)$;
- $\varphi \vdash \bigvee\Phi, \qquad$ if $\varphi \in \Phi$;
- $\varphi(w/v) \vdash \exists v\varphi$;
- $\bigwedge\Phi \vdash \varphi, \qquad$ if $\varphi \in \Phi$;
- $\forall\varphi \vdash \varphi(w/v)$;
- $(\varphi \to_l \psi)\&\varphi \vdash \psi, \qquad \varphi\&(\varphi \to_r \psi) \vdash \psi$;

and is closed under the following rules:

- if $\varphi \vdash \psi$ and $\psi \vdash \rho$, then $\varphi \vdash \rho$;
- if $\varphi \vdash \psi$, then $\varphi\&\rho \vdash \psi\&\rho$ and $\rho\&\varphi \vdash \rho\&\psi$;
- if $\varphi \vdash \psi$ for all $\varphi \in \Phi$, then $\bigvee\Phi \vdash \psi$;
- if $\varphi \vdash \psi$, then $\exists v\varphi \vdash \psi$ provided v does not occur free in ψ.
- if $\psi \vdash \varphi$ for all $\varphi \in \Phi$, then $\psi \vdash \bigwedge\Phi$;
- if $\varphi \vdash \psi$, then $\varphi \vdash \forall v\psi$ provided v does not occur free in φ.
- if $\varphi\&\psi \vdash \rho$, then $\varphi \vdash \psi \to_l \rho$ and $\psi \vdash \varphi \to_r \rho$.

THEOREM 3 (Soundness). *$\varphi \vdash_q \psi$ implies $\varphi \models_q \psi$.*

Proof. For any quantal model $\mathfrak{A}$, the relation $\models^{\mathfrak{A}}$ includes all instances of the above axioms and is closed under the above rules, so it includes $\vdash_q$. Thus $\varphi \vdash_q \psi$ implies $\varphi \models^{\mathfrak{A}} \psi$ for all quantal models $\mathfrak{A}$. ■

3.4 Lindenbaum Models of Fragments

A *fragment* is a *set* $\mathcal{F}$ of formulas that includes all atomic formulas; is closed under the binary connectives $\&$, $\to_l$, $\to_r$, and the quantifiers $\exists v_n$, $\forall v_n$; and is closed under subformulas and variable substitution. Any set Φ of formulas can be enlarged to a fragment: define $\mathcal{F}_0$ by adding all atomic formulas to Φ, and then inductively define $\mathcal{F}_{n+1}$ by closing $\mathcal{F}_n$ under the binary connectives and the quantifiers, and then closing under subformulas and variable-substitution. Then $\mathcal{F} = \bigcup_{n<\omega}\mathcal{F}_n$ is the smallest fragment including Φ.

Definition of the Lindenbaum quantal model $\mathfrak{A}^{\mathcal{F}}$ of a fragment begins with the standard construction. Let $\vdash$ be a relation satisfying all the above axioms and rules. The condition "$\varphi \vdash \psi$ and $\psi \vdash \varphi$" gives an equivalence relation on $\mathcal{F}$. Let $|\varphi|$ be the equivalence class of $\varphi \in \mathcal{F}$ and $S^{\mathcal{F}} = \{|\varphi| : \varphi \in \mathcal{F}\}$. Put $|\varphi| \leq |\psi|$ iff $\varphi \vdash \psi$; $|\varphi| \bullet |\psi| = |\varphi \& \psi|$, $|\varphi| \Rightarrow_l |\psi| = |\varphi \to_l \psi|$ and $|\varphi| \Rightarrow_r |\psi| = |\varphi \to_r \psi|$. The axioms and rules ensure that this yields a well-defined residuated posemigroup $\mathcal{S}^{\mathcal{F}}$ on $S^{\mathcal{F}}$ in which

$$|\exists v_n \varphi| = \textstyle\sum_{p<\omega}|\varphi(v_p/v_n)|, \quad |\forall v_n \varphi| = \prod_{p<\omega}|\varphi(v_p/v_n)| \tag{3.1}$$

$$|\textstyle\bigvee\Phi| = \sum_{\varphi\in\Phi}|\varphi|, \quad \text{when } \bigvee\Phi \in \mathcal{F} \tag{3.2}$$

$$|\textstyle\bigwedge\Phi| = \prod_{\varphi\in\Phi}|\varphi| \quad \text{when } \bigwedge\Phi \in \mathcal{F}. \tag{3.3}$$

The proof of (3.1) is as for finitary first-order logic (e.g. [2, Lemma 3.4.1]), and depends on the fact that a formula has finitely many free variables.

Now put $\mathfrak{A}^{\mathcal{F}} = \langle \mathcal{Q}^{\mathcal{F}}, D, V\rangle$, where $\mathcal{Q}^{\mathcal{F}}$ is the quantale obtained from $\mathcal{S}^{\mathcal{F}}$ by Corollary 2 and having an embedding $f : \mathcal{S}^{\mathcal{F}} \to \mathcal{Q}^{\mathcal{F}}$; D is the set of all variables v_n; and $V(P)(v_{n_1}, \ldots, v_{n_k}) = f|P(v_{n_1}, \ldots, v_{n_k})|$. Then if σ is the D-valuation with $\sigma_n = v_n$, we get

$$\|P(v_{n_1}, \ldots, v_{n_k})\|_\sigma^{\mathfrak{A}^{\mathcal{F}}} = f|P(v_{n_1}, \ldots, v_{n_k})|.$$

We then extend this to show inductively that

$$\|\varphi\|_\sigma^{\mathfrak{A}^{\mathcal{F}}} = f|\varphi| \text{ for all } \varphi \in \mathcal{F}. \tag{3.4}$$

This uses the the definition of $\|\varphi\|_\sigma^{\mathfrak{A}}$, results (3.1)–(3.3), the fact that f preserves the residuated posemigroup operations and any joins and meets existing in $\mathcal{S}^{\mathcal{F}}$; and the general substitutional result that $\|\varphi(v_p/v_n)\|_\sigma^{\mathfrak{A}} = \|\varphi\|_{\sigma(\sigma_p/n)}^{\mathfrak{A}}$ (which holds of any σ in any model $\mathfrak{A}$).

THEOREM 4 (Completeness). *$\varphi \models_q \psi$ implies $\varphi \vdash_q \psi$.*

Proof. Let $\varphi \models_q \psi$, and take a fragment $\mathcal{F}$ containing φ and ψ. Construct $\mathfrak{A}^{\mathcal{F}}$as above using the relation $\vdash_q$ to define $\mathcal{S}^{\mathcal{F}}$. Then $\varphi \models^{\mathfrak{A}^{\mathcal{F}}} \psi$, so $\|\varphi\|_\sigma \leq \|\psi\|_\sigma$ in $\mathfrak{A}^{\mathcal{F}}$ where $\sigma_n = v_n$. Hence $|\varphi| \leq |\psi|$ by (3.4) and the fact that f is order-invariant, so $\varphi \vdash_q \psi$. ∎

4 Covers and Model Structures

We now work with structures $\mathfrak{S} = \langle S, \triangleleft, \cdot, Cov\rangle$, where $\triangleleft$ is a *preorder* (i.e. reflexive and transitive relation); $\cdot$ is an associative operation that is $\triangleleft$-monotone in each argument; and Cov is a function assigning to each $x \in S$ a collection $Cov(x)$ of subsets of S, called the *covers of* x, or x*-covers*.

For $X, Y \subseteq S$, put $X \cdot Y = \{x \cdot y : x \in X \text{ and } y \in Y\}$, $x \cdot Y = \{x\} \cdot Y$ and $X \cdot y = X \cdot \{y\}$. Let $[X) = \{y \in S : (\exists x \in X) x \triangleleft y\}$ and $[x) = [\{x\}) = \{y : x \triangleleft y\}$. X is *increasing* if $[X) \subseteq X$, meaning that if $x \in X$ and $x \triangleleft y$, then $y \in X$.[2] In general, $[X)$ is the smallest increasing superset of X. We write $X \lhd Y$, and say that Y *refines* X, and that X is *refined by* Y, if $(\forall y \in Y)(\exists x \in X) x \lhd y$. Thus $X \lhd Y$ iff $Y \subseteq [X)$. A set X is *cover-closed* if, for all $x \in S$, $(\exists C \in Cov(x))(C \subseteq X)$ implies $x \in X$. A *c-filter* is a set that is increasing and cover-closed.

We call $\mathfrak{S}$ a *model structure*[3] if the following axioms hold for all $x \in S$:

cov1: there exists an x-cover $C \subseteq [x)$;

cov2: if $C \in Cov(x)$ and for all $y \in C$, $C_y \in Cov(y)$, then $\bigcup_{y \in C} C_y \in Cov(x)$.

cov3: if $x \lhd y$, then every x-cover can be refined to a y-cover: $(\forall C \in Cov(x))(\exists B \in Cov(y)) C \lhd B$.

cov4: if $C \in Cov(x)$ and $B \in Cov(y)$, then $C \cdot B$ can be refined to a $x \cdot y$-cover.

cov5: if there exists an x-cover refining $X \cdot Y$, then there exist x', y' with $x' \cdot y' \lhd x$; an x'-cover $X' \subseteq X$; and a y'-cover $Y' \subseteq Y$.

THEOREM 5. *If $\mathfrak{S}$ is a model structure, then $\mathcal{Q}^{\mathfrak{S}} = \langle S^{\mathfrak{S}}, \subseteq, \bullet \rangle$ is a quantale, where $S^{\mathfrak{S}}$ is the set of all c-filters of $\mathfrak{S}$ and $X \bullet Y = [X \cdot Y)$. Joins in $\mathcal{Q}^{\mathfrak{S}}$ are given by $\sum \mathcal{X} = \{x : (\exists C \in Cov(x)) C \subseteq \bigcup \mathcal{X}\}$ for all $\mathcal{X} \subseteq \mathcal{S}^{\mathfrak{S}}$. The residuals of $\bullet$ are given by $X \Rightarrow_l Y = \{z \in S : z \cdot X \subseteq Y\}$ and $X \Rightarrow_r Y = \{z \in S : X \cdot z \subseteq Y\}$.*

Proof. This could be shown by direct set-theoretic reasoning, but more insight into the role of the (cov)-axioms is gained by constructing $\mathcal{Q}^{\mathfrak{S}}$ as the quantale of closed elements of a quantic nucleus. Put $\mathcal{Q}^{\lhd} = \langle S^{\lhd}, \subseteq, \bullet \rangle$ where $S^{\lhd}$ is the set of $\lhd$-increasing subsets of $\mathfrak{S}$ and $X \bullet Y = [X \cdot Y)$. It is readily seen that $\mathcal{Q}^{\lhd}$ is a quantale in which the join of $\mathcal{X} \subseteq S^{\lhd}$ is the set-union $\bigcup \mathcal{X}$ (and the meet is $\bigcap \mathcal{X}$). Note that the definition of $\mathcal{Q}^{\lhd}$ is independent of Cov.

Now define a $\subseteq$-monotonic function j_{Cov} on $S^{\lhd}$ by $j_{Cov}X = \{x \in S : (\exists C \in Cov(x)) C \subseteq X\}$. The axioms (cov1)–(cov4) then ensure that j_{Cov} is a quantic nucleus on $\mathcal{Q}^{\lhd}$, as follows.

First, (cov3) ensures that $j_{Cov}X$ is increasing when X is, for if $x \in j_{Cov}X$ and $x \lhd y$, then there exists $C \in Cov(x)$ with $C \subseteq X$, hence by (cov3) there exists $B \in Cov(y)$ with $B \subseteq [C) \subseteq [X) = X$, implying $y \in j_{Cov}X$.

Next, (cov1) ensures that j_{Cov} is inflationary, for if $x \in X \in S^{\lhd}$, then by (cov1) there exists $C \in Cov(x)$ with $C \subseteq [x) \subseteq X$, implying $x \in j_{Cov}X$. Thus $X \subseteq j_{Cov}X$.

[2] Some treatments of Kripke-Joyal semantics in posets for intuitionistic logic use decreasing sets rather than increasing ones, e.g. [3]. Here we follow the conventions of Kripke's original model theory, as well as of those using preordered groupoids to model other substructural logics [19, 18, 6, 7]. Formally, the distinction is no more than that between a preorder and its converse.

[3] Modal logicians would tend to call this a "frame", but we avoid this term since it has a different meaning in locale theory.

(cov2) ensures that $j_{Cov}j_{Cov}X \subseteq j_{Cov}X$, for if there exists $C \in Cov(x)$ with $C \subseteq j_{Cov}X$, then for all $y \in C$ there exists $C_y \in Cov(y)$ with $C_y \subseteq X$. Then by (cov2), $\bigcup_{y \in Y} C_y$ is an x-cover included in X, showing $x \in j_{Cov}X$.

Finally, (cov4) and (cov3) ensure that j_{Cov} satisfies (2.5). For if $z \in (j_{Cov}X) \bullet (j_{Cov}Y)$, then $x \cdot y \lhd z$ for some x, y such that there is an x-cover $C \subseteq X$ and a y-cover $B \subseteq Y$. By (cov4) there is some $x \cdot y$-cover $A \subseteq [C \cdot B)$, and then by (cov3) there is a z-cover $A' \subseteq [A) \subseteq [C \cdot B) \subseteq [X \cdot Y) = X \bullet Y$, showing that $z \in j_{Cov}(X \bullet Y)$. Hence $(j_{Cov}X) \bullet (j_{Cov}Y) \subseteq j_{Cov}(X \bullet Y)$.

Now for $X \in S^{\lhd}$ we have $j_{Cov}X = X$ iff $j_{Cov}X \subseteq X$ iff X is cover-closed. So the set of j_{Cov}-closed elements of $S^{\lhd}$ is just the set $S^{\mathfrak{S}}$ of c-filters of $\mathfrak{S}$. Thus by the theory explained in Section 2, $\mathcal{Q}^{\mathfrak{S}} = \langle S^{\mathfrak{S}}, \subseteq, \bullet_{j_{Cov}} \rangle$ is a quantale is which $\sum \mathcal{X} = j_{Cov} \bigcup \mathcal{X}$, as required by the statement of this Theorem. But (cov5) ensures that if X and Y are cover-closed, then so is $X \bullet Y$, for if there exists $C \in Cov(x)$ with $C \subseteq X \bullet Y$, then $X \cdot Y \lhd C$, so taking x', y', X', Y' as given by (cov5) we get $x' \in X$ as X is cover-closed, and likewise $y' \in Y$, hence $x \in [X \cdot Y)$ as required because $x' \cdot y' \lhd x$. Thus if X, Y are c-filters, then so is $X \bullet Y$, implying $X \bullet_{j_{Cov}} Y = j_{Cov}(X \bullet Y) = X \bullet Y$. So $\bullet_{j_{Cov}}$ is just $\bullet$ on $S^{\mathfrak{S}}$.

To show the residuals in $\mathcal{Q}^{\mathfrak{S}}$ are as stated, let $W = \{z : z \cdot X \subseteq Y\}$. Then in general $Z \cdot X \subseteq Y$ iff $Z \subseteq W$, so if Y is increasing, $Z \bullet X \subseteq Y$ iff $Z \subseteq W$. But if Y is increasing then so is W by $\lhd$-monotonicity of $\cdot$. Thus if $X, Y \in S^{\lhd}$, then since $W \in S^{\lhd}$ we must have $W = (X \Rightarrow_l Y)$ in $\mathcal{Q}^{\lhd}$. But the left residual of $\bullet_{j_{Cov}}$ in $\mathcal{Q}^{\mathfrak{S}}$ is just the restriction of the left residual of $\bullet$ in $\mathcal{Q}^{\lhd}$ to $S^{\mathfrak{S}}$, so if $X, Y \in S^{\mathfrak{S}}$, then $W = (X \Rightarrow_l Y)$ in $\mathcal{Q}^{\mathfrak{S}}$. Similarly for $X \Rightarrow_r Y$. ■

The covering concept comes of course from topology, where an open cover of a set x is any collection C of open sets whose union includes x. A property holds *locally* of x if it holds of all members of some cover of x. For example, a function is *locally constant* on x if it is constant on each member of some x-cover. If we take S to be the set of all open subsets of some topological space and put $x \lhd y$ iff $y \subseteq x$, $x \cdot y = x \cap y$ and $C \in Cov(x)$ iff $x \subseteq \bigcup C$, then we obtain a model structure (in which "refines" has its usual meaning for topological covers). Indeed this construction works in any quantale, as we now show.

THEOREM 6. *Every quantale $\mathcal{Q}$ is isomorphic to the quantale $\mathcal{Q}^{\mathfrak{S}}$ of some model structure $\mathfrak{S}$.*

Proof. Let $\mathcal{Q} = \langle Q, \leq, \bullet \rangle$. Define $\mathfrak{S} = \langle S, \lhd, \cdot, Cov \rangle$ by putting $S = Q$; $x \lhd y$ iff $y \leq x$; $x \cdot y = x \bullet y$; and $C \in Cov(x)$ iff $x \leq \sum C$ for all $C \subseteq Q$. Then $\lhd$ is a preorder and $\cdot$ is associative and $\lhd$-monotone. Moreover, $[x) = (x\downarrow)$.

Thus a set X is increasing in $\mathfrak{S}$ iff it is $\leq$-*decreasing* in $\mathcal{Q}$, i.e. $x \in X$ implies $(x\downarrow) \subseteq X$. In particular, $x\downarrow$ is increasing, and is also cover-closed, for if $C \in Cov(y)$ and $C \subseteq (x\downarrow)$, then $y \leq \sum C \leq x$, hence $y \in (x\downarrow)$. Moreover, if X is any c-filter and we put $x = \sum X$, then $X \in Cov(x)$ and so $x \in X$ by cover-closure, hence $X = (x\downarrow)$ as X is $\leq$-decreasing.

Thus the map $x \mapsto (x{\downarrow})$ is a bijection between Q and the set $S^{\mathfrak{S}}$ of c-filters of $\mathfrak{S}$. This map is order-invariant: $x \leq y$ iff $(x{\downarrow}) \subseteq (y{\downarrow})$. But it is readily seen that $(x \bullet y){\downarrow} = [(x{\downarrow}) \cdot (y{\downarrow}))$ so altogether the map is an isomorphism between $\mathcal{Q}$ and the quantale $\mathcal{Q}^{\mathfrak{S}}$ of the previous Theorem. It remains to show $\mathfrak{S}$ is a model structure.

(cov1): Any C with $x \in C \subseteq [x)$ will do.

(cov2): If $x \leq \sum C$, and $(\forall y \in C)(y \leq \sum C_y)$, then $x \leq \sum_{y \in C}(\sum C_y) = \sum(\bigcup_{y \in C} C_y)$.

(cov3): If $C \in \mathit{Cov}(x)$ and $x \lhd y$, then $y \leq x \leq \sum C$, so just take $B = C$.

(cov4): If $C \in \mathit{Cov}(x)$ and $B \in \mathit{Cov}(y)$, then $x \cdot y \leq (\sum C) \cdot (\sum B) = \sum(C \cdot B)$, with the equality following from the distributive laws (2.1) and (2.2). Thus $C \cdot B$ is itself an $x \cdot y$-cover.

(cov5): Suppose there is an x-cover $C \subseteq [X \cdot Y)$. Put $X' = \{a \in X : (\exists b \in Y)(\exists c \in C) a \cdot b \lhd c\} \subseteq X$ and $Y' = \{b \in Y : (\exists a \in X)(\exists c \in C) a \cdot b \lhd c\} \subseteq Y$. Let $x' = \sum X'$ and $y' = \sum Y'$, so that $X' \in \mathit{Cov}(x')$ and $Y' \in \mathit{Cov}(y')$. It remains to show $x' \cdot y' \lhd x$. Now if $c \in C$, then $a \cdot b \lhd c$ for some $a \in X$ and $b \in Y$. Then $a \in X'$ and $b \in Y'$, so $c \leq a \cdot b \leq x' \cdot y'$. Hence $x \leq \sum C \leq x' \cdot y'$, as required. ∎

The axioms (cov1)–(cov5) are almost minimal requirements for $\mathcal{Q}^{\mathfrak{S}}$ to be a quantale in Theorem 5. (cov5) could be weakened as it is only needed when X and Y are c-filters. In that case we could add the requirement that every cover is an increasing set, since both Theorems 5 and 6 would hold under this requirement. As it stands, the model structure defined in Theorem 6 satisfies several other strengthenings and additional conditions. It has $[x) \in \mathit{Cov}(x)$ and $\{x\} \in \mathit{Cov}(x)$, each of which implies (cov1). For (cov3) it has the stronger property that if $x \lhd y$, then every x-cover is a y-cover. For (cov4) it has the conclusion that $C \cdot B$ is an $x \cdot y$ cover when $C \in \mathit{Cov}(x)$ and $B \in \mathit{Cov}(y)$. It even has the property that $\mathit{Cov}(x)$ is closed under supersets. But it does not have the property that *every x-cover is a subset of* $[x)$, which is a basic assumption in the definition of coverings on posets in locale theory [10, 3] and is fundamental to the categorical view of coverings in a Grothendieck topology on a category. In the $\mathfrak{S}$ of Theorem 6, this property would require that an x-cover have $x = \sum C$, rather than $x \leq \sum C$. This makes (cov3) problematic, although the other (cov)-axioms still hold. If $\mathcal{Q}$ is a *locale*, then (cov3) does hold in this case, for if $y \leq x = \sum C$, then the subset $\{y \sqcap c : c \in C\}$ of $[y)$ is a refinement of C whose join is y, by the distribution of the lattice meet $\sqcap$ over $\sum$. But that argument is not available in a general quantale.

5 Kripke-Joyal Semantics

A *structural model* for the language of Section 3 has the form $\mathcal{M} = \langle \mathfrak{S}, D, V \rangle$, where $\mathfrak{S}$ is a model structure; D is a set of individuals; and for each k-ary predicate letter P, $V(P)$ is a function assigning a c-filter of $\mathfrak{S}$ to each k-tuple of elements of D. In other words, $V(P) : D^k \to S^{\mathfrak{S}}$, where $S^{\mathfrak{S}}$ is the set of all c-filters of $\mathfrak{S}$. From such an $\mathcal{M}$ we immediately obtain the quantal model $\mathfrak{A}^{\mathcal{M}} = \langle \mathcal{Q}^{\mathfrak{S}}, D, V \rangle$, where $\mathcal{Q}^{\mathfrak{S}} = \langle S^{\mathfrak{S}}, \subseteq, \bullet \rangle$ is the quantale of Theorem 5. We define $\varphi \models^{\mathcal{M}} \psi$ to mean that $\varphi \models^{\mathfrak{A}^{\mathcal{M}}} \psi$. Also we write $\|\varphi\|_\sigma^{\mathcal{M}}$ for the value $\|\varphi\|_\sigma^{\mathfrak{A}^{\mathcal{M}}}$ given by the general definition of $\|\varphi\|_\sigma^{\mathfrak{A}}$ from Section 3. Thus $\|\varphi\|_\sigma^{\mathcal{M}}$ is a c-filter, and

$$\varphi \models^{\mathcal{M}} \psi \quad \text{iff} \quad \|\varphi\|_\sigma^{\mathcal{M}} \subseteq \|\psi\|_\sigma^{\mathcal{M}} \text{ for all } D\text{-valuations } \sigma. \tag{5.1}$$

In the converse direction, from any quantal model $\mathfrak{A} = \langle \mathcal{Q}, D, V \rangle$ we obtain the structural model $\mathcal{M}^{\mathfrak{A}} = \langle \mathfrak{S}, D, V^{\mathfrak{A}} \rangle$, where $\mathfrak{S}$ is the model structure of Theorem 6 for which there is an isomorphism of quantales $f : \mathcal{Q} \cong \mathcal{Q}^{\mathfrak{S}}$, and $V^{\mathcal{M}}(P) = f \circ V(P) : D^k \to S^{\mathfrak{S}}$.

Since the isomorphism f preserves all the operations involved in the definition of $\|\varphi\|_\sigma^{\mathfrak{A}}$, an inductive proof then shows that in general $f \circ \|\varphi\|_\sigma^{\mathfrak{A}} = \|\varphi\|_\sigma^{\mathcal{M}^{\mathfrak{A}}}$, and so $\|\varphi\|_\sigma^{\mathfrak{A}} \leq \|\psi\|_\sigma^{\mathfrak{A}}$ iff $\|\varphi\|_\sigma^{\mathcal{M}^{\mathfrak{A}}} \subseteq \|\psi\|_\sigma^{\mathcal{M}^{\mathfrak{A}}}$. Hence

$$\varphi \models^{\mathfrak{A}} \psi \quad \text{iff} \quad \varphi \models^{\mathcal{M}^{\mathfrak{A}}} \psi. \tag{5.2}$$

THEOREM 7 (Completeness for Structural Models). *For all formulas φ and ψ the following are equivalent:*

(1) $\varphi \vdash_q \psi$.

(2) $\varphi \models^{\mathcal{M}} \psi$ *for all structural models* $\mathcal{M}$.

Proof. (1) implies (2): by the Soundness Theorem 3, $\varphi \vdash_q \psi$ implies $\varphi \models^{\mathfrak{A}^{\mathcal{M}}} \psi$. (2) implies (1): if $\varphi \nvdash_q \psi$, then by the Completeness Theorem 4, $\varphi \not\models^{\mathfrak{A}} \psi$ for some quantal model $\mathfrak{A}$. But then $\varphi \not\models^{\mathcal{M}^{\mathfrak{A}}} \psi$ by (5.2). ■

In a model structure $\mathcal{M} = \langle S, \triangleleft, \cdot, Cov, D, V \rangle$, a *satisfaction relation* can be defined by using the notation

$$\mathcal{M}, x \models \varphi[\sigma]$$

to mean that $x \in \|\varphi\|_\sigma^{\mathcal{M}}$. This can be read "$\varphi$ is true/satisfied in $\mathcal{M}$ at x under σ". Thus (5.1) becomes

$$\varphi \models^{\mathcal{M}} \psi \quad \text{iff} \quad \text{for all } x \text{ and } \sigma,\ \mathcal{M}, x \models \varphi[\sigma] \text{ implies } \mathcal{M}, x \models \psi[\sigma].$$

A purely model theoretic description of this satisfaction relation in a structural model can be derived by applying the description of the quantale operations of $\mathcal{Q}^{\mathfrak{S}}$ given in Theorem 5 to the general definition of $\|\varphi\|_\sigma^{\mathfrak{A}}$. This is given below (with the symbol $\mathcal{M}$ suppressed as it is constant throughout). The cases of $\bigvee$, $\bigwedge$ and $\exists$ are just like the corresponding clauses in the Kripke-Joyal semantics of models in sheaf categories [11, Theorem VI.7.1].

$$
\begin{array}{lll}
x \models P(v_{n_1},\dots,v_{n_k})[\sigma] & \text{iff} & x \in V(P)(\sigma_{n_1},\dots,\sigma_{n_k}).\\
x \models \varphi \& \psi[\sigma] & \text{iff} & \text{for some } y \text{ and } z \text{ such that } y\cdot z \lhd x,\\
 & & y \models \varphi[\sigma] \text{ and } z \models \psi[\sigma].\\
x \models \varphi \to_l \psi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } x\cdot y \models \psi[\sigma].\\
x \models \varphi \to_r \psi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } y\cdot x \models \psi[\sigma].\\
x \models \bigvee\Phi[\sigma] & \text{iff} & \text{there is an } x\text{-cover } C \text{ such that for all } z \in C,\\
 & & z \models \varphi[\sigma] \text{ for some } \varphi \in \Phi.\\
x \models \bigwedge\Phi[\sigma] & \text{iff} & x \models \varphi[\sigma] \text{ for all } \varphi \in \Phi.\\
x \models \exists v_n\varphi[\sigma] & \text{iff} & \text{there is an } x\text{-cover } C \text{ such that for all } z \in C,\\
 & & z \models \varphi[\sigma(d/n)] \text{ for some } d \in D.\\
x \models \forall v_n\varphi[\sigma] & \text{iff} & x \models \varphi[\sigma(d/n)] \text{ for all } d \in D.
\end{array}
$$

In place of the last clause, the sheaf semantics (and Kripke's intuitionistic semantics) typically has

$$x \models \forall v_n\varphi[\sigma] \quad \text{iff} \quad x \lhd y \text{ implies } y \models \varphi[\sigma(d/n)] \text{ for all } d \in D.$$

But this follows from the last clause because the c-filter $\|\varphi[\sigma(d/n)]\|_\sigma^{\mathcal{M}}$ is $\lhd$-increasing. The semantics of $\to_l$ is sometimes given in the form

$$x \models \varphi \to_l \psi[\sigma] \quad \text{iff} \quad (x\cdot y \lhd z \text{ and } y \models \varphi[\sigma]) \text{ implies } z \models \psi[\sigma],$$

but this follows from the above because $\|\psi\|_\sigma^{\mathcal{M}}$ is $\lhd$-increasing. Similarly for $\varphi \to_r \psi$.

If we take the *classical* disjunction of Φ to be a formula that is satisfied at x precisely when some member of Φ is satisfied at x, then the above criterion for $x \models \bigvee\Phi$ is that the classical disjunction of Φ is *locally* satisfied at x, i.e. satisfied throughout some cover of x. Similarly, the criterion for $x \models \exists v_n\varphi$ is that the classical existential quantification of φ is locally satisfied at x.

6 Modalities

We now take a few steps in the direction of modal logic over quantales. A *modal operator* on a quantal or other poset will be taken to be any unary function j that is monotone. This can be used to give algebraic semantics to a new unary connective (modality) ∇ by defining $\|\nabla\varphi\| = j\|\varphi\|$. Structurally ∇ can be interpreted by adding to the definition of a model structure $\mathfrak{S}$ a new binary relation $\prec$ on S, and requiring that in a model based on $\mathfrak{S}$,

$$\mathcal{M}, x \models \nabla\varphi[\sigma] \quad \text{iff} \quad \text{for some } y,\ x \prec y \text{ and } \mathcal{M}, y \models \varphi[\sigma]. \tag{6.1}$$

The relation $\prec$ induces the modal operator $j_\prec$ on $\langle \mathcal{P}S, \subseteq\rangle$ having

$$j_\prec X = \{x \in S : \exists y(x \prec y \in X)\}. \tag{6.2}$$

Then the definition $\|\nabla\varphi\|_\sigma^{\mathcal{M}} = j_\prec\|\varphi\|_\sigma^{\mathcal{M}}$ ensures that (6.1) holds. But we also want to ensure that $j_\prec X$ is a c-filter whenever X is. This can be achieved by requiring that a model structure satisfies the following two conditions

$$\text{if } x \lhd y \text{ and } x \prec z\text{, then for some } w,\ z \lhd w \text{ and } y \prec w. \tag{6.3}$$

cov7: if there exists an x-cover included in $j_{\prec} X$, then there exists a y with $x \prec y$, and a y-cover included in X.

(6.3) states that $(\rhd \circ \prec) \subseteq (\prec \circ \rhd)$, where $\rhd$ is the converse of $\lhd$, and suffices to make $j_{\prec} X$ $\lhd$-increasing if X is. (cov7) makes $j_{\prec} X$ cover-closed if X is. Then the quantale $\mathcal{Q}^{\mathfrak{S}}$ of c-filters is closed under $j_{\prec}$.

In the opposite direction, starting with a modal operator j on a quantale $\mathcal{Q}$, define a relation $\prec_j$ on the structure $\mathfrak{S}$ of Theorem 6 having $\mathcal{Q} \cong \mathcal{Q}^{\mathfrak{S}}$, by putting $x \prec_j y$ iff $x \leq jy$. This in turn induces the operator $j_{\prec_j}$ on $\mathcal{Q}^{\mathfrak{S}}$.

THEOREM 8. *$\prec_j$ satisfies* (6.3) *and* (cov7)*, and the isomorphism $x \mapsto (x{\downarrow})$ from $\mathcal{Q}$ to $\mathcal{Q}^{\mathfrak{S}}$ preserves the operators j and $j_{\prec_j}$.*

Proof. For (6.3), if $x \lhd y$ and $x \prec_j z$, then $y \leq x \leq jz$, so $y \prec_j z$, hence (6.3) holds in the strong form that we can take $w = z$.

For (cov7), suppose there exists $C \in \mathit{Cov}(x)$ with $C \subseteq j_{\prec_j} X$. Hence $x \leq \sum C$. Let $B = \{z \in X : \exists c \in C(c \prec_j z)\}$, and put $y = \sum B$. Then $B \in \mathit{Cov}(y)$ and $B \subseteq X$, so it remains to show $x \prec_j y$. But if $c \in C$, then $c \in j_{\prec_j} X$, so there exists z with $c \prec_j z \in X$. Then $c \leq jz$ and $z \in B$, so $z \leq y$, hence $jz \leq jy$ as j is monotone, and thus $c \leq jy$. Therefore $x \leq \sum C \leq jy$, implying $x \prec_j y$.

Preservation of the operators requires that $(jx){\downarrow} = j_{\prec_j}(x{\downarrow})$. But if $z \in (jx){\downarrow}$, then $z \prec_j x \in (x{\downarrow})$, so $z \in j_{\prec_j}(x{\downarrow})$ by (6.2). And if $z \in j_{\prec_j}(x{\downarrow})$, then $z \prec_j y \leq x$ for some y, so $z \leq jy \leq jx$, giving $z \in (jx){\downarrow}$. ■

We can now consider correspondences between properties of j and properties of $\prec$ in a manner familiar from modal logic. If $\prec$ is reflexive, then $j_{\prec}$ is inflationary, and if $\prec$ is transitive, then $j_{\prec} j_{\prec} X \subseteq j_{\prec} X$. Hence if $\prec$ is a preorder, $j_{\prec}$ is a *closure operator* (inflationary and idempotent). Conversely, if j is an inflationary modal operator on a quantale, then $\prec_j$ is reflexive, and if $j \circ j \leq j$ (i.e. $\forall x(jjx \leq jx)$), then $\prec_j$ is transitive. Thus

a closure operator on a quantale can be represented as the operator $j_{\prec}$ induced on the quantale of c-filters of a model structure by a preorder relation $\prec$.

Recall from (2.5) that a quantic nucleus is a closure operator satisfying $jx \bullet jy \leq j(x \bullet y)$. For $j_{\prec}$ to satisfy (2.5) it suffices that

$$\text{if } x \cdot y \lhd z,\ x \prec x' \text{ and } y \prec y'\text{, then } \exists z'(z \prec z' \text{ and } x' \cdot y' \lhd z'). \tag{6.4}$$

Conversely, if j is a quantic nucleus, then (6.4) holds in the strong form that we can replace its conclusion by $x' \cdot y' \lhd z$. For if $z \leq x \bullet y$, $x \leq jx'$

and $y \leq jy'$, then $z \leq jx' \bullet jy' \leq j(x' \bullet y')$. So any quantic nucleus can be represented as the operator induced by a preorder satisfying this strong form of (6.4).

To axiomatise the logic characterised by the class of quantal models with a modality, we just add the rule

$$\varphi \vdash \psi \text{ implies } \nabla\varphi \vdash \nabla\psi$$

to the proof theory of Section 3.3. Then putting $j|\varphi| = |\nabla\varphi|$ gives a well-defined modal operator on the Lindenbaum posemigroup $\mathcal{S}^{\mathcal{F}}$ of a fragment as in Section 3.4. Viewing $\mathcal{S}^{\mathcal{F}}$ as a subalgebra of the quantale $\mathcal{Q}^{\mathcal{F}}$, we can lift j to an operator j^+ on $\mathcal{Q}^{\mathcal{F}}$ by putting

$$j^+x = \sum\{ja \in \mathcal{S}^{\mathcal{F}} : a \leq x\}.$$

j^+ agrees with j when restricted to $\mathcal{S}^{\mathcal{F}}$, and is monotone so can be represented as the operator induced by $\prec_{j^+}$.

If the axiom $\varphi \vdash \nabla\varphi$ is added, then j is inflationary, hence so is j^+ (proof: $x = \sum\{a \in \mathcal{S}^{\mathcal{F}} : a \leq x\} \leq \sum\{ja : a \leq x\}$). The axiom $\nabla\nabla\varphi \vdash \nabla\varphi$ enforces $j \circ j \leq j$, but this appears only to lift to j^+ if j^+ preserves joins, yielding $j^+j^+x = \sum\{jja : a \leq x\}$. On the other hand the nucleus condition (2.5) does lift to j^+, as shown by the following calculation, in which $a, b, c \in \mathcal{S}^{\mathcal{F}}$:

$$\left(\sum_{a \leq x} ja\right) \bullet \left(\sum_{b \leq y} jb\right) = \sum_{a \leq x, b \leq y} (ja \bullet jb) \leq \sum_{a \leq x, b \leq y} j(a \bullet b) \leq \sum_{c \leq x \bullet y} jc.$$

Here the equality is given by the quantale distribution laws (2.1), (2.2), the first inequality by (2.5) for j, and the second by $a \bullet b \leq x \bullet y$.

To give a completeness theorem for the logic of closure operators on quantales, a different lifting of j to $\mathcal{Q}^{\mathcal{F}}$ can be used, namely

$$j^{\#}x = \prod\{ja : a \in \mathcal{S}^{\mathcal{F}} \text{ and } x \leq ja\}.$$

Interestingly, this $j^{\#}$ is a closure operator for any j whatsoever, but its restriction to $\mathcal{S}^{\mathcal{F}}$ agrees with j (equivalently, the embedding of $\mathcal{S}^{\mathcal{F}}$ into $\mathcal{Q}^{\mathcal{F}}$ preserves j and $j^{\#}$) precisely when j itself is a closure operator, facts that the reader may like to verify.

On the other hand it does not appear that (2.5) lifts to $j^{\#}$ in general, so an axiomatisation of the modal logic of quantic nuclei on quantales awaits further investigation.

7 Conclusion and Further Work

The aim of this paper has been to extend the idea of Kripke-Joyal semantics to a generalisation of the intuitionistic context, and a minimal syntax has been used for this purpose. There are many possible additions and extensions that could be considered, and further questions that suggest themselves. On the syntactic side we have not considered the modelling of individual constants, function symbols, the equality predicate, or true and false propositional constants $\top$ and $\bot$ (although for the latter we could use $\bigvee\emptyset$ and $\bigwedge\emptyset$). Semantically we used a kind of "constant domain" model, with a single set D forming the range of quantifiable variables. It may be of interest to explore the variable-domain approach of Kripke's intuitionistic models, in which each element x of a model structure has its own domain D_x that is used in evaluating quantifiers at x.

Proof-theoretically we used simple sequents with a single formula on each side of the symbol $\vdash$, so there is scope for discussion of something more like the usual Gentzen calculi. There are additional axioms required for the various kinds of quantale that have been considered. It is often assumed that a quantale is *unital*, i.e. there is an identity element for $\bullet$, and this requires a new propositional constant E, and a distinguished c-filter in model structures to serve as $\|\mathsf{E}\|^{\mathcal{M}}$ and be the identity element in $\mathcal{Q}^{\mathfrak{S}}$. Then there are quantales that are right-sided ($\varphi\&\top \vdash \varphi$), left-sided ($\top\&\varphi \vdash \varphi$), and idempotent ($\varphi\&\varphi \dashv\vdash \varphi$). The combination of axioms $\varphi\&\psi \vdash \varphi$, $\varphi\&\psi \vdash \psi$ and $\varphi \vdash \varphi\&\varphi$ force $a \bullet b$ to be the meet of a and b, hence $\|\varphi\&\psi\|^{\mathcal{M}} = \|\varphi\|^{\mathcal{M}} \cap \|\psi\|^{\mathcal{M}} = \|\varphi \wedge \psi\|^{\mathcal{M}}$.

The *involutary* quantales have an involution a^* that preserves joins and reverses the arguments of $\bullet$, and amongst these the *Hilbert* quantales have an orthocomplement $a^\perp$ that obeys De Morgan's laws in relation to $\bigvee$ and $\bigwedge$. The enhancement of our model structures to interpret unary connectives corresponding to a^* and $a^\perp$ is a natural topic for investigation.

As to modal logic, a quantic nucleus j seems rather schizophrenic as a modal operator. Its properties as a closure operator suggest it should model a "diamond" modality with the bounded-existential modelling condition of (6.1), as we have done here. But in locales the nucleus property (2.5) implies that $j(a \bullet b)$ is the lattice meet of ja and jb, which is a property more reminiscent of the "box" modalities that have the bounded-universal semantics

$$\mathcal{M}, x \models \nabla\varphi[\sigma] \quad \text{iff} \quad \text{for all } y,\ x \prec y \text{ implies } \mathcal{M}, x \models \varphi[\sigma]. \tag{7.1}$$

Indeed in [8] we gave a Kripke semantics for the finitary modal logic of nuclei on locales, using this kind of semantics. So there is more to be explored here, including the general study of modalities fulfilling (7.1), regardless of whether they interpret quantic nuclei.

There are intimate connections between cover systems and quantic nuclei. From any preordered semigroup $\langle S, \triangleleft, \cdot\rangle$ we obtain a quantale $\langle S^\triangleleft, \subseteq, \bullet\rangle$ in which $S^\triangleleft$ is the set of $\triangleleft$-increasing subsets, $X \bullet Y = [X \cdot Y)$, and joins are set-unions. We saw in the proof of Theorem 5 that from any system *Cov* on S satisfying (cov1)–(cov4) we get a quantic nucleus j_{Cov} on $S^\triangleleft$ by putting $j_{Cov}(X) = \{x \in S : (\exists C \in Cov(x))C \subseteq X\}$. But conversely, from any quantic nucleus j on $S^\triangleleft$ we get a system Cov_j satisfying (cov1)–(cov4) by putting $Cov_j(x) = \{C \in S^\triangleleft : x \in j(C)\}$. Then $j_{Cov_j} = j$. If $Cov(x)$ is always a subset of $S^\triangleleft$ closed under supersets in $S^\triangleleft$ (as indeed $Cov_j(x)$ is), then $Cov_{j_{Cov}} = Cov$. These relationships, and the general theory of cover systems for quantales, warrant further investigation.

Finally we raise the question of the separate study of quantal *geometric* formulas, which are those formed from atomic ones using only &, $\bigvee$ and $\exists$. It would be of interest to axiomatise the class of sequents of geometric formulas that are valid in quantal models in the sense of (5.1). Our completeness method depended on the the presence of the residuals $\Rightarrow_l$ and $\Rightarrow_r$ in a Lindenbaum posemigroup $\mathcal{S}^{\mathcal{F}}$ to construct the quantale $\mathcal{Q}^{\mathcal{F}}$ by the method of Corollary 2. It is not clear that the approach would work for Lindenbaum algebras generated by a language without $\rightarrow_l$ and $\rightarrow_r$. It may be necessary to use more general sequents of lists of formulas to achieve this. An alternative approach to completeness might be to use the standard Henkin method to build "canonical" models whose points are sets of formulas with syntactic closure properties that mimic those of the "truth sets" $\{\varphi : \mathfrak{A}, x \models \varphi[\sigma]\}$ defined by points in models under valuations. Of course the construction of such canonical models would be of interest in general, and not just for geometric formulas.

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