Quantification over names and modalities

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ABSTRACT. Making quantification and modalities interact properly has raised numerous issues that have been tackled in various ways. We propose an alternative approach with quantifiers ranging over names (closed terms) rather than individuals. To be general enough and not restricted to dealing exclusively with Henkin or Herbrand structures, we distinguish between the language used to describe a structure, and the language used to talk about a structure. We assume that the former is rich enough to provide designators for all individuals, while the latter might be a subset of the former, allowing one to make some individuals "unspeakable of," as is needed for the developments of classical logic. These considerations are applied to Kripke semantics, and classical results—the Löwenheim Skolem and compactness theorems—are revisited. With quantification over names, the results are more general and the proofs simpler, being based on restrictions rather than on extensions of the original vocabulary.

1 Introduction

Providing a suitable semantics for first-order modal logic has proved to be a challenging enterprise. The issues have been extensively discussed in [2], and a solution proposed. The crux of the matter is how to let quantifiers interact with modalities. In a formula of the form $\exists x \Box \psi(x)$, some object o is selected in the current possible world M, and asserted to have property ψ in all possible worlds accessible from M. But o might not exist in some of those worlds. In the constant domain semantics with rigid designators, the matter is simple. It is also simple with the generalization provided by intuitionistic Kripke frames, where the domain of a possible world that is accessible from M has to contain the domain of M (see [7] for a recent use of these frames). But for arbitrary domains and interpretations of the function symbols that vary from possible world to possible world, the issues are numerous. To resolve these issues, distinctions between existence of objects and designation of objects have been carefully unravelled ([2]), or an 'actually' operator has been introduced ([11]). In this paper we take another path, and suggest that quantification could range over names rather than individuals. This is not a new proposal. It goes back to the work of Ruth Barcan-Marcus [8], and has been embraced or investigated by many researchers. See for instance [1, 3, 4, 5, 6] for various developments. But when quantification over names is used, it is assumed that the domains of the intended interpretations are such that all individuals have names, or even that every individual has a unique name. This assumption is suitable in some contexts, like when dealing with the Herbrand semantics of logic programs, but inappropriate for classical semantics.

If we consider canonical models (structures all of whose individuals interpret a closed term), that arise from Henkin's proof of the completeness of first-order logic (see *e.g.* [10]), it is clear that models where all objects have a name play a natural role even in the realm of classical logic. They are obtained by *enriching* the initial vocabulary with enough constants to provide witnesses for existential formulas. Why not proceed in the opposite way, by starting with a language that is rich enough to denote all individuals in any intended interpretation, and then *restricting* the vocabulary so that some individuals cannot be named, if required? In other words, distinguish between the language used to describe a structure and the language used to talk about a structure. Canonical models are particularly interesting for another reason: they identify individuals with the set of all names for this individual. This concept is clear enough when only one canonical structure is involved, but what if many canonical structures are considered at once? Then we might encounter sets of names that are distinct but have a nonempty intersection. The implications of this observation suggest to consider, besides the usual mathematical structures, more abstract, 'linguistic' structures that consist of a set of constraints to denote objects and interpret predicates, and use the more abstract structures as the components of a Kripke frame.

We proceed as follows. In Section 2 we develop these considerations in more detail. In Section 3, we provide the formal machinery and express the natural relationships between mathematical structures, built from sets of individuals, and abstract structures, that generalize Herbrand structures and can be thought of as sets of linguistic constraints in the use of a set of names. In Section 4 we let abstract structures be the components of a Kripke frame, yielding abstract Kripke frames. We then put classical Kripke frames in relation with abstract Kripke frames, and propose a semantics based on this relationship. In Section 5 we show how the Löwenheim–Skolem and compactness results can be reformulated in the context of abstract Kripke frames. We conclude in Section 6.

2 Two kinds of structures

From now on $\mathcal{V}$ denotes a *vocabulary*, that is, a set of function and predicate symbols. For reasons that will soon become clear, we depart from the usual practice of including only nonlogical predicate symbols in $\mathcal{V}$, and we require that the distinguished predicate symbol $=$ be a member of $\mathcal{V}$ when dealing with a language with equality, and that $=$ not be a member of $\mathcal{V}$ when dealing with a language without equality. In this section, a *formula* over $\mathcal{V}$ can refer to a member of $\mathcal{L}_{\omega\omega}(\mathcal{V})$ (the set of first-order formulas over $\mathcal{V}$), or to a member of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ (the extension of $\mathcal{L}_{\omega\omega}(\mathcal{V})$ that accepts countable disjunctions or conjunctions of statements), the latter set playing a role in the second part of the paper. Recall that a mathematical structure M over

$\mathcal{V}$ is a pair, with the first element of the pair being the domain of M, denoted $|M|$, and the second element of the pair being the set of interpretations in M of the function and predicate symbols in $\mathcal{V}$.

The usual proof of the completeness theorem for classical first-order logic is based on the construction (Henkin's construction) of a special mathematical structure whose domain is a partition of the set of closed terms over some extension V of the original vocabulary, with the members of a given individual being viewed as the possible names for that individual; such a mathematical structure, which is said to be *Henkin*, can then be simply represented as a pair consisting of a partition of the set of closed terms over V and a set of closed atoms over V. The *Herbrand structures* considered in Logic programming are a particular kind of mathematical structure over a vocabulary V that contains at least one constant, where every individual interprets a unique closed term, viewed as the unique name of that individual, and is usually identified with it; a Herbrand structure can then be simply represented as a set of closed atoms over V (this set being referred to as the *Herbrand base*), and identified with a Henkin structure whose individuals are singletons.

Take a partition U of the set of closed terms over $\mathcal{V}$ and a set of B of closed atoms over $\mathcal{V}$. As we have reminded, the pair (U, B) is usually identified with a (Henkin) mathematical structure, but we can take a different perspective. Rather than identifying (U, B) with a mathematical representation of some form of reality, we can view (U, B) as a purely syntactic, linguistic object, a blueprint from which mathematical representations of some forms of reality can be obtained. More precisely, given a vocabulary $\mathcal{V}^\star \subseteq \mathcal{V}$, the pair (U, B) can act as a blueprint for any mathematical representation of some form of reality, or for any mathematical structure M over $\mathcal{V}^\star$ that is *described* by (U, B) in the following sense:

- any individual ι of M is given at least one name, *i.e*, is associated with at least one closed term over $\mathcal{V}$, with a unique member of U being the set of names given to ι;
- a closed term t over $\mathcal{V}^\star$ is one of the names given to the unique individual of M that interprets t;
- a closed term of the form $f(t_1, \dots, t_n)$ is such that if individuals $\iota_1, \dots, \iota_n$ have respective names $t_1, \dots, t_n$, then $f(t_1, \dots, t_n)$ is a name of the individual $f^M(\iota_1, \dots, \iota_n)$;
- a closed atom over $\mathcal{V}^\star$ of the form $p(t_1, \dots, t_n)$ belongs to B iff the individuals of M denoted by $t_1, \dots, t_n$ are in relation p^M in M.

From this perspective, a closed formula φ over $\mathcal{V}$ cannot be true in (U, B)—that notion is meaningless if (U, B) does not mathematically represent some form of reality, but just offers a blueprint for such a representation: φ can only be believed or *accepted* in (U, B), in the sense that for any reality and for any representation of that reality in the form of a mathematical structure M over $\mathcal{V}$, if M is described by (U, B) then φ is true in M. We will refer to (U, B), viewed as we have just described, as an *abstract structure* over $\mathcal{V}$. The distinction is futile in classical first-order logic, and in Logic programming for the particular case of Herbrand mathematical structures. We claim that it is beneficial to make the distinction in Kripke semantics. This is because in Kripke semantics, the basic notion of truth involves a whole set of structures rather than a unique one. To make this point clearer, assume that the only function symbols in $\mathcal{V}$ are four constants a, b, c, d, and consider the following sets:

$$U_1 = \{\{a, b\}, \{c, d\}\} \quad U_2 = \{\{a\}, \{b\}, \{c, d\}\} \quad U_3 = \{\{a\}, \{b, c\}, \{d\}\}.$$

First consider three mathematical structures M_1, M_2 and M_3 over $\mathcal{V}$ with respective domains U_1, U_2 and U_3, such that for all $M \in \{M_1, M_2, M_3\}$, $\iota \in |M|$ and $x \in \{a, b, c, d\}$, ι is the interpretation of x in M iff $x \in \iota$. So M_1, M_2 and M_3 are Henkin mathematical structures over $\mathcal{V}$. The view that the individual $\{a, b\}$ of M_1 has two names a and b, the individual $\{a\}$ of M_2 has one name a, etc., is not problematic as long as M_1, M_2 and M_3 are considered independently. But consider M_1, M_2 and M_3 together. Since the sets $\{a, b\}$, $\{a\}$ and $\{b\}$ are three distinct mathematical objects, they are different individuals. How can we then have the absolute view that a and b are the names of $\{a, b\}$, and also that a is the name of $\{a\}$ and b the name of $\{b\}$? How can different objects share common names? How can two constants be the two names of some individual, and also the respective names of two distinct individuals? It is conceptually much simpler to assume that a and b are not the names of the individual $\{a, b\}$ of M_1, that a is not the name of the individual $\{a\}$ of M_2, etc: as far as the activity of naming individuals is concerned, the individuals $\{a, b\}$ and $\{a\}$ are no more special than the numbers 2 and π: all are mathematical objects. It is conceptually much simpler to assume that in some contexts, some agent names the individual $\{a, b\}$ of M_1 as either a or b, in some contexts, some agent names the individual $\{a\}$ of M_2 as a, and in some contexts, some agent names the individual $\{b\}$ of M_3 as b. So the point is that it is fine to consider M_1, M_2 and M_3 independently and view $\{a, b\}$ and $\{c, d\}$, or $\{a\}$, $\{b\}$ and $\{c, d\}$, or $\{a\}$, $\{b, c\}$ and $\{d\}$ both as individuals and sets of names. But it is difficult to keep that absolute view and consider M_1, M_2 and M_3 together, without relativizing and externalizing the notion of "being the name of an individual." This is precisely what can be achieved with the notion of an *abstract structure* describing a mathematical structure, as we now explain.

When dealing with abstract structures over $\mathcal{V}$, we conceive of the members of $\mathcal{V}$ as names of functions and relations. But when dealing with mathematical structures over $\mathcal{V}$, we conceive of the members of $\mathcal{V}$ as just placeholders to refer to unnamed functions and relations. This means that "$\{c, d\}$ is the interpretation of c in M_1" only says that some nullary function in M_1 exists and maps $\emptyset$ to the individual $\{c, d\}$. To better express our intentions, we could use distinct kinds of vocabularies to define mathematical structures and abstract structures. The vocabularies involved in the definition of mathematical structures could be sets of symbols like

$$\{-_0^{f,0}, -_1^{f,0}, -_2^{f,2}, -_3^{p,1}, -_4^{p,1}\}$$

with $-_0^{f,0}$ and $-_1^{f,0}$ being placeholders for two nullary functions, $-_2^{f,2}$ being a placeholder for a binary function, and $-_3^{p,1}$ and $-_4^{p,1}$ being two placeholders for two unary predicates. These placeholders, being devoid of any meaning, would make it easier to conceptualize that we are only pointing to mathematical objects—functions, relations—but not naming them. Vocabularies of the form that we commonly use would only be involved in the definition of abstract structures, to name from the outside the mathematical objects that make up the domain of a mathematical structure.

3 Abstract structures

Now for the formal details. The notion of abstract structure is defined in two steps:[1]

DEFINITION 1. A *labeling partition over* $\mathcal{V}$ is a partition X of the set of closed terms over $\mathcal{V}$ with the following properties.

- If $\mathcal{V}$ does not contain equality then all members of X are singletons.
- Let $n \in \mathbb{N}$, n-ary function symbol f in $\mathcal{V}$, and closed terms $t_1, \ldots, t_n$, $t'_1, \ldots, t'_n$ over $\mathcal{V}$ be such that for all non-null $i \leq n$, t_i and t'_i belong to the same member of X. Then $f(t_1, \ldots, t_n)$ and $f(t'_1, \ldots, t'_n)$ also belong to the same member of X.

The first clause in Definition 1 allows one a smooth treatment of languages with or without equality. Note that if $\mathcal{V}$ contains no constant then $\emptyset$ is the unique labeling partition over $\mathcal{V}$.

DEFINITION 2. An *abstract structure* (*over* $\mathcal{V}$) is a pair $\mathfrak{M}$ consisting of:

- a labeling partition over $\mathcal{V}$, called the *labeling partition of* $\mathfrak{M}$, whose members are referred to as *labeling classes* (*of* $\mathfrak{M}$);
- a set of closed atoms over $\mathcal{V}$, called the *acceptance set of* $\mathfrak{M}$, having the property that for all $n \in \mathbb{N}$, n-ary predicate symbols p in $\mathcal{V}$ distinct from equality, and closed terms $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ over $\mathcal{V}$, if
  - $p(t_1, \ldots, t_n)$ belongs to the acceptance set of $\mathfrak{M}$, and
  - for all nonull $i \leq n$, t_i and t'_i belong to the same labeling class

  then $p(t'_1, \ldots, t'_n)$ belongs to the acceptance set of $\mathfrak{M}$.

So given an abstract structure $\mathfrak{M}$, $\mathfrak{M}(0)$ is the labeling partition of $\mathfrak{M}$ and $\mathfrak{M}(1)$ is the acceptance set of $\mathfrak{M}$.

[1] $\mathbb{N}$ denotes the set of natural numbers.

The correspondence between mathematical structures and abstract structures is defined as expected.

DEFINITION 3. Let an abstract structure $\mathfrak{M}$ over $\mathcal{V}$ be given.

Let a vocabulary $\mathcal{V}^\star \subseteq \mathcal{V}$ and a mathematical structure M over $\mathcal{V}^\star$ be given. A *homomorphism from* $\mathfrak{M}$ *into* M is a surjective mapping h from $\bigcup \mathfrak{M}(0)$ into the domain of M, with the following properties.

- For all closed terms t_1, t_2 over $\mathcal{V}$, $h(t_1) = h(t_2)$ iff t_1 and t_2 belong to the same member of $\mathfrak{M}(0)$.
- For all $n \in \mathbb{N}$, n-ary function symbols f in $\mathcal{V}^\star$, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$, $h(f(t_1, \ldots, t_n)) = f^M(h(t_1), \ldots, h(t_n))$.
- For all $n \in \mathbb{N}$, n-ary predicate symbols p in $\mathcal{V}^\star$ distinct from $=$, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$, $p(t_1, \ldots, t_n)$ belongs to $\mathfrak{M}(1)$ iff $(h(t_1), \ldots, h(t_n))$ belongs to p^M.

Given a vocabulary $\mathcal{V}^\star$ and a mathematical structure M over $\mathcal{V}^\star$, we say that $\mathfrak{M}$ *describes* M iff $\mathcal{V}^\star \subseteq \mathcal{V}$ and a homomorphism from $\mathfrak{M}$ into M exists.

For instance, assume that $\mathcal{V}$ is defined as in Section 2, and recall the mathematical structures M_1, M_2 and M_3 and the sets U_1, U_2 and U_3 defined in Section 2. Consider three abstract structures $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ over $\mathcal{V}$ of respective form (U_1, B_1), (U_2, B_2) and (U_3, B_3). Then for all $i \in \{1, 2, 3\}$, M_i could be described by $\mathfrak{M}_i$, but it could not be described by $\mathfrak{M}_j$ for $j \in \{1, 2, 3\} \setminus \{i\}$ (whether M_i is actually described by $\mathfrak{M}_i$ depends of course on B_i and on how the predicate symbols in $\mathcal{V}$ are interpreted in M_i). Each of $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ could describe other mathematical structures over $\mathcal{V}$. For instance, $\mathfrak{M}_1$ could describe a mathematical structure over $\mathcal{V}$ whose domain is U_1 where both a and b are interpreted by $\{c, d\}$ and both c and d are interpreted by $\{a, b\}$; $\mathfrak{M}_2$ could describe a mathematical structure over $\mathcal{V}$ whose domain is U_3 where a is interpreted by $\{a\}$, b is interpreted by $\{b, c\}$ and both c and d are interpreted by $\{d\}$; $\mathfrak{M}_1$ could describe a mathematical structure over $\mathcal{V}$ whose domain is $\{2, 4\}$, where both a and b are interpreted by 2 and both c and d are interpreted by 4. Moreover, each of $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ could describe mathematical structures over a vocabulary strictly included in $\mathcal{V}$. For instance, let $\mathcal{V}^\star$ be the union of $\{a, b, d\}$ with the set of predicate symbols in $\mathcal{V}$. Then $\mathfrak{M}_2$ could describe a mathematical structure over $\mathcal{V}^\star$ whose domain is U_2 where a is interpreted by $\{a\}$, b is interpreted by $\{b\}$, and d is interpreted by $\{c, d\}$; also, $\mathfrak{M}_2$ could describe a mathematical structure over $\mathcal{V}^\star$ whose domain is U_3 where a is interpreted by $\{a\}$, b is interpreted by $\{b, c\}$, and d is interpreted by $\{d\}$. More generally, an abstract structure over $\mathcal{V}$ can describe many mathematical structures over a vocabulary that is included in $\mathcal{V}$, and a mathematical structure over a vocabulary that is included in $\mathcal{V}$ can be described by many abstract structures over $\mathcal{V}$.

The next property states that all mathematical structures can be represented by abstract structures. Though not every individual of a mathematical structure M over $\mathcal{V}$ has a name built from the function symbols in $\mathcal{V}$, every individual of M can still be assumed to have a name that is possibly "hidden,", *i.e.*, not expressible with $\mathcal{V}$. Distinguishing between the vocabulary that describes a mathematical structure and the vocabulary that is used to make formulas about it is the key to defining the semantics of classical logic on the basis of abstract structures rather than on the basis of mathematical structures.

PROPERTY 4. Assume that $\mathcal{V}$ contains equality. Let a mathematical structure M over $\mathcal{V}$ be given. Let F be a set of function symbols not in $\mathcal{V}$ such that the set of closed terms over $\mathcal{V} \cup F$ of the form $f(t_1, \ldots, t_n)$, where $n \in \mathbb{N}$ and f is an n-ary function symbol in F of arity n, has cardinality at least equal to the cardinality of M. Then M is described by at least one abstract structure over $\mathcal{V} \cup F$.

COROLLARY 5. *Assume that $\mathcal{V}$ contains equality. Let a mathematical structure M over $\mathcal{V}$ be given. Let F be a set of constants not in $\mathcal{V}$ of cardinality at least equal to the cardinality of M. Then M is described by at least one abstract structure over $\mathcal{V} \cup F$.*

The previous property and corollary do not generalize to vocabularies that do not contain equality. This is because if $\mathcal{V}$ does not contain equality, then for every mathematical structure M over $\mathcal{V}$ that is described by an abstract structure over $\mathcal{V}$, two distinct closed terms over $\mathcal{V}$ have to be interpreted in M by two distinct individuals. This is at no real loss of generality, and it is technically convenient.

Given an abstract structure $\mathfrak{M}$ over $\mathcal{V}$ and a closed member φ of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$, we can define the notion $\mathfrak{M} \models \varphi$ either directly from Definition 2, or indirectly from the relationship between formal and mathematical structures. Directly:

DEFINITION 6. Let an abstract structure $\mathfrak{M}$ over $\mathcal{V}$ be given. Given a closed member φ of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$, the notion "$\varphi$ is accepted in $\mathfrak{M}$," or "$\mathfrak{M}$ is a model of φ," denoted $\mathfrak{M} \models \varphi$, is inductively defined as follows.

- Suppose that $\mathcal{V}$ contains equality, and let closed terms t_1, t_2 over $\mathcal{V}$ be given. Then:
  - $\mathfrak{M} \models t_1 = t_2$ if t_1 are t_2 belong to the same labeling class of $\mathfrak{M}$;
  - $\mathfrak{M} \models t_1 \neq t_2$ if t_1 are t_2 belong to distinct labeling classes of $\mathfrak{M}$.
- Let $n \in \mathbb{N}$, an n-ary predicate symbol p in $\mathcal{V}$, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$ be given. Then:
  - $\mathfrak{M} \models p(t_1, \ldots, t_n)$ if $p(t_1, \ldots, t_n) \in \mathfrak{M}(1)$;
  - $\mathfrak{M} \models \neg p(t_1, \ldots, t_n)$ if $p(t_1, \ldots, t_n) \notin \mathfrak{M}(1)$.
- Let a set X of closed members of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ be given. Then:
  - $\mathfrak{M} \models \bigvee X$ iff there exists $\psi \in X$ such that $\mathfrak{M} \models \psi$;
  - $\mathfrak{M} \models \bigwedge X$ iff for all $\psi \in X$, $\mathfrak{M} \models \psi$.

- Let $\psi \in \mathcal{L}_{\omega_1\omega}(\mathcal{V})$ and variable x be such that $\mathrm{fv}(\psi) = \{x\}$. Then:
  - $\mathfrak{M} \models \exists x\psi$ iff $\mathfrak{M} \models \psi[t/x]$ for some closed term t over $\mathcal{V}$;
  - $\mathfrak{M} \models \forall x\psi$ iff $\mathfrak{M} \models \psi[t/x]$ for all closed terms t over $\mathcal{V}$.

Indirectly:

PROPERTY 7. For all abstract structures $\mathfrak{M}$ over $\mathcal{V}$, vocabularies $\mathcal{V}^\star \subseteq \mathcal{V}$ and closed members φ of $\mathcal{L}_{\omega_1\omega}(\mathcal{V}^\star)$, the following conditions are equivalent.

- $\mathfrak{M} \models \varphi$.
- $\mathfrak{M}$ describes some mathematical structure M over $\mathcal{V}^\star$ with $M \models \varphi$.
- For all mathematical structures M over $\mathcal{V}^\star$ that $\mathfrak{M}$ describes, $M \models \varphi$.

We write $\mathfrak{M} \not\models \varphi$ if φ is not accepted in $\mathfrak{M}$. Also, we will use the notation $\mathfrak{M} \models T$ and $\mathfrak{M} \not\models T$ for subsets T of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$.

Recall that if $\mathcal{V}$ does not contain equality then the labeling classes of a abstract structure over $\mathcal{V}$ are singletons. This constraint does not have any implication on the notion of a formula being accepted in an abstract structure:

LEMMA 8. *Assume that $\mathcal{V}$ does not contain equality. Let an abstract structure $\mathfrak{M}$ over $\mathcal{V} \cup \{=\}$ and a closed $\varphi \in \mathcal{L}_{\omega_1\omega}(\mathcal{V})$ be given. Then φ is accepted in $\mathfrak{M}$ iff φ is accepted in the (unique) abstract structure over $\mathcal{V}$ that has the same acceptance set as $\mathfrak{M}$.*

COROLLARY 9. *Assume that $\mathcal{V}$ does not contain equality, and let φ be a closed member of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$. Then φ is accepted in an abstract structure over $\mathcal{V}$ iff φ is accepted in an abstract structure over $\mathcal{V} \cup \{=\}$.*

The choice of "accepted" in Definition 6 is more convenient than dictated by any epistemic considerations. It emphasizes that the relationship between a formula and an abstract structure can be kept at the linguistic level, with no need to refer to a mathematical model of some form of reality. At a formal level, the differences and connections between abstract structures and mathematical structures are straightforward. The formalization that supports these differences and connections would remain a rather sterile exercise if it was not utilized in a modal framework, where in contrast to the classical framework of first-order logic, structures are considered in relation to each other. The key point is in the range of the variables. Variables range over individuals w.r.t. mathematical structures, but they should be viewed as ranging over names (not labeling classes!), w.r.t. abstract structures. Let a closed member of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ of the form $\exists x\varphi$ be given. If M is a mathematical structure over $\mathcal{V}$, the intended interpretation of $M \models \exists x\varphi$ is that one of M's individuals has property φ. If $\mathfrak{M}$ is an abstract structure over $\mathcal{V}$, the intended interpretation of $\mathfrak{M} \models \exists x\varphi$ is that for some name (some closed term over $\mathcal{V}$) t, the formula $\varphi[t/x]$ is accepted in $\mathfrak{M}$, meaning that for all mathematical structures M over $\mathcal{V}$, if M is described by $\mathfrak{M}$ then the individual of M which is named t by $\mathfrak{M}$, has property φ. This is crucially important when dealing simultaneously with mathematical structures that have arbitrary domains: a variable that denotes a particular individual u in a particular mathematical structure is comparable to a pointer that has been assigned the value u when viewed from a mathematical structure whose domain contains u, but it is comparable to a void pointer when viewed from any other mathematical structure. On the other hand, a variable that denotes a particular name of a particular abstract structure $\mathfrak{M}$ can point to any individual of any mathematical structure M that can be described by $\mathfrak{M}$: it suffices to let $\mathfrak{M}$ properly describe M.

Most logical notions can be defined on the basis of abstract structures; formally, mathematical structures are not really needed. Mathematical structures can be viewed as lower level interpretations, closer to reality. More precisely, mathematical structures provide a way to reinterpret the notion that a closed member of $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ is accepted in an abstract structure over $\mathcal{V}$, in terms of the classical notion of truth.

4 Abstract Kripke frames

Abstract structures provide a real benefit in a modal-theoretic context. It suffices to let abstract structures rather than mathematical structures play the role of possible worlds in Kripke semantics. More precisely:

DEFINITION 10. An *abstract Kripke frame over* $\mathcal{V}$ is a pair (K, R) where K is a set of abstract structures over $\mathcal{V}$, and R is a binary relation over K whose domain is K.

An abstract Kripke frame (K, R) is said to have *constant labeling* iff all members of K have the same labeling partition.

We impose the weak restriction that the domain of R is equal to K in order to simplify the formal developments, but that assumption is otherwise inessential. Of course, the notion of an abstract Kripke frame having a constant labeling is the counterpart to the notion of a Kripke frame having constant domain.

To be precise enough, let us define the language we will be using in the rest of the paper.

DEFINITION 11. The set $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ of (*infinitary*) *modal formulas over* $\mathcal{V}$ is inductively defined as follows.

- All atoms over $\mathcal{V}$ and their negations belong to $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$.
- For all countable $X \subseteq \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, $\bigvee X$ and $\bigwedge X$ belong to $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$.
- For all variables x and for all $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, $\exists x\varphi$ and $\forall x\varphi$ belong to $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$.
- For all $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, $\Diamond\varphi$ and $\Box\varphi$ both belong to $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$.

If disjunctions and conjunctions are applied to finite sets only, then the resulting set of (finite) modal formulas is denoted $\mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$.

In the definition of both $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ and $\mathcal{L}_{\omega\omega}(\mathcal{V})$, negation is assumed to be applicable to atomic formulas only, which amounts to imposing a negation normal form (to negate arbitrary modal formulas, it suffices to inductively define an operator $\sim$ with $\sim\Box\varphi = \Diamond{\sim}\varphi$, $\sim\Diamond\varphi = \Box{\sim}\varphi$, $\sim\exists x\varphi = \forall x{\sim}\varphi$, $\sim\forall x\varphi = \exists x{\sim}\varphi$, $\sim\neg\varphi = \varphi$, and for atomic φ's, $\sim\varphi = \neg\varphi$). The usual syntax on finite formulas is introduced as abbreviations: $\varphi_1 \vee \varphi_2$ for $\bigvee\{\varphi_1, \varphi_2\}$, $\varphi_1 \wedge \varphi_2$ for $\bigwedge\{\varphi_1, \varphi_2\}$, etc. These decisions on the primitive syntax result in no loss in generality, but simplify the formal developments. The set of all variables that occur free in a modal formula φ is denoted $\mathrm{fv}(\varphi)$.

Given an abstract Kripke frame $\mathcal{K} = (K, R)$ and a member $\mathfrak{M}$ of K, the notion of a closed modal formula φ being *accepted in* $(\mathcal{K}, \mathfrak{M})$, or of $(\mathcal{K}, \mathfrak{M})$ being a *model* of φ, denoted $(\mathcal{K}, \mathfrak{M}) \Vdash \varphi$, is defined in the usual way from Definition 6, the key clauses still being those that deal with the quantifiers. Of course, we have to accept to use the same vocabulary to define all the abstract structures involved in a given abstract Kripke frame.

We see the benefit of variables ranging over closed terms rather than individuals, and of Kripke frames defined from abstract structures rather than from mathematical structures. We avoid some issues that have to be faced in order to interpret in the context of a (nonabstract) Kripke frame any closed formula that contains a nonclosed subformula of the form $\Diamond\psi$ or $\Box\psi$. In particular, both following issues are avoided, where $\mathcal{K} = (K, R)$ denotes a (nonabstract) Kripke frame and M a member of K.

First issue: A natural intended interpretation of $\exists x\Diamond\psi$ in $(\mathcal{K}, M)$ is that some individual ι in the domain of M has property $\Diamond\psi$ in $(\mathcal{K}, M)$. But for all $N \in K$ with $R(M, N)$, the domain of N does not have to include the domain of M, hence might not contain ι.

Second issue: Assume that the domain of N is included in the domain of M for every member N of K with $R(M, N)$. Then the previous issue disappears. Now let ι in the domain of M be such that ι has property $\Diamond\psi$ of x in $(\mathcal{K}, M)$, and as a consequence, ι has property ψ of x in $(\mathcal{K}, N)$, for some member N of K with $R(M, N)$. Suppose that $\mathcal{V}$ contains a constant c, and that ι is the interpretation of c in M. A natural syntactic operation is the substitution operation that transforms $\Diamond\psi$ into $\Diamond\psi[c/x]$. But $\Diamond\psi[c/x]$ does not necessarily mean that the individual denoted by c in N has property ψ of x in $(\mathcal{K}, N)$. Indeed, it might be that c does not denote ι in N. In that case, substituting in $\Diamond\psi$ all free occurrences of x by c "twists" the meaning of $\Diamond\psi$.

Let us relate abstract Kripke frames to mathematical (classical) Kripke frames. We can generalize Definition 3 as follows.

DEFINITION 12. Let an abstract Kripke frame $\mathcal{K} = (K, R)$ over $\mathcal{V}$, a vocabulary $\mathcal{V}^\star \subseteq \mathcal{V}$, and a mathematical Kripke frame $\mathcal{K}^\star = (K^\star, R^\star)$ over $\mathcal{V}^\star$ be given.

A *homomorphism from* $\mathcal{K}$ *into* $\mathcal{K}^\star$ is a bijective mapping h from K into $K^\star$ with the following properties.

- For all $\mathfrak{M} \in K$, $\mathfrak{M}$ describes $h(\mathfrak{M})$.
- For all $\mathfrak{M}, \mathfrak{N} \in K$, $R^\star(h(\mathfrak{M}), h(\mathfrak{N}))$ holds iff $R(\mathfrak{M}, \mathfrak{N})$ holds.

A *full Kripke frame* (*based on* $\mathcal{K}$) is a pair of the form $(K^\star, R^\star, h)$ where h is a homomorphism from $\mathcal{K}$ into $\mathcal{K}^\star$.

Note that Property 7 does not generalize from structures to full Kripke frames. The semantics of modal logic can be defined w.r.t. a mathematical Kripke frame and one of its linguistic blueprints, with different linguistic blueprints for the same mathematical Kripke frame possibly yielding different logical notions. More precisely:

DEFINITION 13. Let an abstract Kripke frame $\mathcal{K}$ over $\mathcal{V}$, a vocabulary $\mathcal{V}^\star \subseteq \mathcal{V}$, a mathematical Kripke frame $\mathcal{K}^\star = (K^\star, R^\star)$ over $\mathcal{V}^\star$, and a homomorphism h from $\mathcal{K}$ into $\mathcal{K}^\star$ be given. Let $M \in K^\star$ and $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V}^\star)$ be given. We write $(\mathcal{K}^\star, h, M) \Vdash \varphi$ if $(K, h^{-1}(M)) \Vdash \varphi$.

Let us illustrate the previous definition with a few examples. We can write that for all $M \in K^\star$,

$$(\dagger) \qquad (\mathcal{K}^\star, h, M) \Vdash \forall x \Box \exists y (x = y)$$

because for all $\mathfrak{M} \in K$, $(\mathfrak{M}, \mathcal{K})$ is a model of $\forall x \Box \exists y(x = y)$. In other words, we do not read ($\dagger$) as: for all $N \in K^\star$ with $R^\star(M, N)$, the domain of M is included in the domain of N. We read ($\dagger$) as: w.r.t. the chosen linguistic blueprint $\mathcal{K}$ for $K^\star$, every name that can be used to make statements about M can also be used to make statements about any $N \in K^\star$ with $R^\star(M, N)$ (actually, about any $N \in K^\star$). Note that with this example, the linguistic blueprint for $\mathcal{K}^\star$ can be chosen arbitrarily. Now assume that $K^\star$ consists of two mathematical structures M and N over $\mathcal{V}^\star$, with the domain of N being strictly larger than the domain of M. Then

$$(\ddagger) \qquad (K^\star, h, M) \Vdash \exists x \exists y((x = y) \wedge \Diamond(x \neq y)),$$

in the sense that if $\mathfrak{M}$ is the (unique) member of K that describes M then $(\mathfrak{M}, \mathcal{K})$ is a model of $\exists x \exists y((x = y) \wedge \Diamond(x \neq y))$. In other words, we read ($\ddagger$) as: w.r.t. the chosen linguistic blueprint $\mathcal{K}$ for $\mathcal{K}^\star$, there exists two names that denote the same individual in M but different individuals in N. Note that with this example again, the linguistic blueprint for $\mathcal{K}^\star$ can be chosen arbitrarily. Finally, assume that $\mathcal{V}^\star$ contains no constant and $K^\star$ consists of two mathematical structures M and N over $\mathcal{V}^\star$, with the domain of N being at most as large as the domain of M. Then

$$(K^\star, h, M) \Vdash \exists x \exists y((x = y) \wedge \Diamond(x \neq y)).$$

for some choices of h, whereas

$$(K^\star, h, M) \nVdash \exists x \exists y((x = y) \wedge \Diamond(x \neq y)).$$

for other choices of h. Indeed, there exists a linguistic blueprint for $\mathcal{K}^\star$ in two names denote the same individual in M but distinct individuals in N, and there exists a linguistic blueprint for $\mathcal{K}^\star$ in which no two names denote the same individual in M but distinct individuals in N.

To conclude, with quantification over names, we can develop modal logic either on the basis of abstract Kripke frames, or on the basis of mathematical (classical) Kripke frames together with a homomorphism from an abstract Kripke frame into the latter, that represents a set of underlying linguistic blueprints—with different sets of linguistic blueprints possibly resulting in different semantics.

5 Classical results revisited

The classical statement of the Löwenheim–Skolem theorem is: given a set X of closed finite formulas over $\mathcal{V}$, if all members of X are true in a mathematical structure over $\mathcal{V}$ then all members of X are true in a mathematical structure over $\mathcal{V}$ whose domain is countable. If we develop a logical framework on the basis of abstract structures rather than on the basis of mathematical structures, the Löwenheim–Skolem theorem needs to be reformulated. A natural reformulation is: given a set X of closed finite formulas over $\mathcal{V}$, if all members of X are accepted in an abstract structure over $\mathcal{V}$, then all members of X are accepted in an abstract structure over $\mathcal{V}^\star$, for some countable subset $\mathcal{V}^\star$ of $\mathcal{V}$ that contains all symbols which occur in φ. This claim is indeed correct, and will be a particular case of one of the main results to be proved in this section.

The Löwenheim–Skolem theorem has been generalized to $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$. Let X be a set of closed infinitary formulas over $\mathcal{V}$. It is not the case that if all members of X are true in a mathematical structure over $\mathcal{V}$, then all members of X are necessarily true in a mathematical structure over $\mathcal{V}$ whose domain is countable. But $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ still enjoys a weak form of the Löwenheim–Skolem theorem in the sense that if X is countable, then the previous claim becomes correct. In the realm of abstract structures, this translates to: if X is a countable set of closed infinitary formulas over $\mathcal{V}$ and if all members of X are accepted in an abstract structure over $\mathcal{V}$, then all members of X are necessarily accepted in an abstract structure over $\mathcal{V}^\star$ for some countable subset $\mathcal{V}^\star$ of $\mathcal{V}$ that contains all symbols which occur in X. This will also be a particular case of one of the main results to be proved in this section. Of course, if X is countable then $\bigwedge X$ belongs to $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$, and the claims about $\mathcal{L}_{\omega_1\omega}(\mathcal{V})$ are better expressed in terms of closed infinitary formulas rather than in terms of countable sets of closed infinitary formulas.

In this section, we show how to express the Löwenheim–Skolem theorem for $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ and both the class of all abstract Kripke frames over $\mathcal{V}$ and the class of all abstract Kripke frames over $\mathcal{V}$ that have constant labeling. Countability can then also be imposed on the number of abstract structures that make up the abstract Kripke frame. This revisiting of the Löwenheim–Skolem theorem will allow one to easily derive the corresponding versions of the compactness theorem for $\mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$. These results can be derived from a common set of arguments, based on a variation of the usual argument based on the classical notions of tableaux and Hintikka sets. One of the interesting aspects of the revisited proof is that we need to restrict vocabularies rather than restricting domains. This means that Henkin's construction has to be changed. Rather than *enriching* the underlying vocabulary with a countable set of constants, suitable to define a mathematical structure over the extended vocabulary where every individual has a name, and providing potential witnesses for all existential formulas, we will *reduce* the underlying vocabulary, still making sure that enough function symbols remain so that potential witnesses for all existential formulas can be provided. This is achieved in the lemma below, based on the following definition that uses a form of *term subsumption*.

DEFINITION 14. Let a $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ and a closed term t over $\mathcal{V}$ be given. We say that t is *unrestricted in* φ iff for all terms t' over $\mathcal{V}$ and for all substitutions ν, if t' occurs in φ and $t = t'\nu$ then t' is a variable.

Intuitively, if t is unrestricted in φ then φ cannot make a statement that is specific to t: if φ is used to make a statement about t then for any closed term t', φ can also be used to make the same statement about t'. For instance, assume that $\mathcal{V}$ consists of a constant a, a unary function symbol s, a binary function symbol g, and a unary predicate symbol P.

- No closed term over $\mathcal{V}$ is unrestricted in

  $$P(g(a,y)) \wedge P(g(g(x,y), g(x, s(s(s(x))))))$$

  as every closed term over $\mathcal{V}$ is an instance of a, $s(x)$, or $g(x,y)$.

- a is the only closed term over $\mathcal{V}$ that is unrestricted in

  $$P(x) \wedge P(s(s(y))) \wedge P(g(y,x))$$

  as x and y are the only terms t that occur in the formula above and such that a is an instance of t.

- No closed term over $\mathcal{V}$ is unrestricted in

  $$\bigwedge\{P(g(g(a,x),x_1)), P(g(g(s(x),x_0),x_1)),\\ P(g(g(g(a,x),x_1),x_2)), P(g(g(g(s(x),x_0),x_1),x_2))\ldots\}$$

  as can be easily verified.

- Infinitely many closed terms over $\mathcal{V}$ are unrestricted in

  $$P(s(s(a))) \wedge P(g(x,y)),$$

  for instance, all terms of the form $s^n(a)$ with n (the number of successive applications of s) at least equal to 3.

- Infinitely many closed terms over $\mathcal{V}$ are unrestricted in
$$P(g(a,y)) \wedge P(g(s(x),y)) \wedge P(g(x,x)),$$
for instance, all terms of the form $g(g(a,a),s^n(a))$.

LEMMA 15. *Let a modal formula φ over $\mathcal{V}$, a variable x, an abstract Kripke frame $\mathcal{K} = (K, R)$ over $\mathcal{V}$, and a member $\mathfrak{M}$ of K be such that* $\mathrm{fv}(\varphi) = \{x\}$ *and $(\mathcal{K}, \mathfrak{M}) \Vdash \exists x\varphi$. Assume that infinitely many closed terms over $\mathcal{V}$ are unrestricted in φ. Let a closed term t over $\mathcal{V}$ be unrestricted in φ. Then there exists an abstract Kripke frame $\mathcal{K}^\star = (K^\star, R^\star)$ over $\mathcal{V}$ and a member $\mathfrak{M}^\star$ of $K^\star$ with the following properties.*

- $(\mathcal{K}^\star, \mathfrak{M}^\star) \Vdash \varphi[t/x]$.
- *If $\mathcal{K}$ has constant labeling then $\mathcal{K}^\star$ has constant labeling.*

Proof. Let X denote the set of closed terms over $\mathcal{V}$ that are unrestricted in φ. Let a closed term v over $\mathcal{V}$ be such that $(\mathcal{K}, \mathfrak{M}) \Vdash \varphi[v/x]$. Let $u \in X$ be given. Let h be a bijection from $X \setminus \{u\}$ into X. Let h^* be the unique total mapping from the set of closed terms over $\mathcal{V}$ into the set of closed terms over $\mathcal{V}$ that satisfies the following conditions.

- For all $t \in X \setminus \{u\}$, $h^\star(t) = h(t)$.
- $h^\star(u) = v$.
- For all $n \in \mathbb{N}$, n-ary function symbols f in $\mathcal{V}$, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$, if $f(t_1, \ldots, t_n) \notin X$ then $h^\star(f(t_1, \ldots, t_n))$ is equal to $f(h^\star(t_1), \ldots, h^\star(t_n))$.

Now for all $\mathfrak{N} \in K$, let $\mathfrak{N}^\star$ be the unique structure over $\mathcal{V}$ that satisfies the following conditions.

- Assume that $\mathcal{V}$ contains equality. Let closed terms t_1, t_2 over $\mathcal{V}$ be given. Then t_1 and t_2 belong to the same member of $\mathfrak{N}^\star(0)$ iff $h^\star(t_1)$ and $h^\star(t_2)$ belong to the same member of $\mathfrak{N}(0)$.
- Let an $n \in \mathbb{N}$, an n-ary predicate symbol p over $\mathcal{V}$ distinct from equality, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$ be given. Then $p(t_1, \ldots, t_n)$ belongs to $\mathfrak{N}^\star(1)$ iff $p(h^\star(t_1), \ldots, h^\star(t_n))$ belongs to $\mathfrak{N}(1)$.

Put $K^\star = \{\mathfrak{N}^\star \mid \mathfrak{N} \in K\}$. Let $R^\star$ be the accessibility relation on $K^\star$ such that for all $\mathfrak{S}_1, \mathfrak{S}_2 \in K^\star$, $R^\star(\mathfrak{S}_1, \mathfrak{S}_2)$ holds iff there exists $\mathfrak{N}_1, \mathfrak{N}_2 \in K$ such that $\mathfrak{S}_1 = \mathfrak{N}_1^\star$, $\mathfrak{S}_2 = \mathfrak{N}_2^\star$, and $R(\mathfrak{N}_1, \mathfrak{N}_2)$ holds. It is easy to verify that $\mathcal{K}^\star$ has constant labeling if $\mathcal{K}$ has constant labeling. To complete the proof of the proposition, it suffices to show the following.

> $(*)$ Let a subformula ψ of φ, $n \in \mathbb{N}$, distinct variables $x_1, \ldots, x_n$, closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$, and a member $\mathfrak{M}$ of K be such that $\mathrm{fv}(\psi) = \{x_1, \ldots, x_n\}$. Then the formula $\psi[t_1/x_1, \ldots, t_n/x_n]$ is accepted in $(\mathcal{K}, \mathfrak{M})$ iff the formula $\psi[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n]$ is accepted in $(\mathcal{K}^\star, \mathfrak{M}^\star)$.

Let $m, n \in \mathbb{N}$, m-ary predicate symbol p in $\mathcal{V}$, distinct variables $x_1, \ldots, x_n$, closed terms $t_1, \ldots, t_n$, and terms $t'_1, \ldots, t'_m$ be such that $p(t'_1, \ldots, t'_m)$ is a subformula of φ and $x_1, \ldots, x_n$ are the variables that occur in at least one of $t'_1, \ldots, t'_m$. Then

$$p(h^\star(t'_1[t_1/x_1, \ldots, t_n/x_n]), \ldots, h^\star(t'_m[t_1/x_1, \ldots, t_n/x_n])) = \\ p(t'_1[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n], \ldots, t'_m[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n]).$$

Since

$$p(t'_1, \ldots, t'_m)[t_1/x_1, \ldots, t_n/x_n] = \\ p(t'_1[t_1/x_1, \ldots, t_n/x_n], \ldots, t'_m[t_1/x_1, \ldots, t_n/x_n])$$

and

$$p(t'_1, \ldots, t'_m)[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n] = \\ p(t'_1[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n], \ldots, t'_m[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n])$$

it follows that for all members $\mathfrak{M}$ of K, the following are equivalent.

- $p(t'_1, \ldots, t'_m)[t_1/x_1, \ldots, t_n/x_n]$ is accepted in $\mathfrak{M}$;
- $p(t'_1, \ldots, t'_m)[h^\star(t_1)/x_1, \ldots, h^\star(t_n)/x_n]$ is accepted in $\mathfrak{M}^\star$.

With the fact that $h^\star$ is onto the set of closed terms over $\mathcal{V}$, this implies that for all $\mathfrak{N}_1, \mathfrak{N}_2 \in K$, $R^\star(\mathfrak{N}_1^\star, \mathfrak{N}_2^\star)$ holds iff $R(\mathfrak{N}_1, \mathfrak{N}_2)$ holds. Using the fact that $h^\star$ is onto the set of closed terms over $\mathcal{V}$ again, it is then easy to verify $(*)$ by induction on the complexity of the subformulas of φ. ■

The classical definition of Hintikka sets is easily extended to the syntax of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$. Very standard techniques yield technical details, definitions, lemmas and propositions—provided in the appendix—that allow one to derive the following form of the Löwenheim–Skolem and compactness theorems.

PROPOSITION 16. *Let $\mathcal{C}$ be either the set of abstract Kripke frames over $\mathcal{V}$, or the set of abstract Kripke frames over $\mathcal{V}$ that have constant labeling.*

1. *Let a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ be satisfiable in $\mathcal{C}$. Then there exists a countable subset $\mathcal{V}^\star$ of $\mathcal{V}$, an abstract Kripke frame $\mathcal{K} = (K, R)$ over $\mathcal{V}^\star$, and a member $\mathfrak{M}$ of K such that $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V}^\star)$, $(\mathcal{K}, \mathfrak{M}) \Vdash \varphi$, and:*
   - *$\mathcal{K}$ has constant labeling if all members of $\mathcal{C}$ have constant labeling;*
   - *K is countable.*
2. *For all countable $T \subseteq \mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$, if all finite subsets of T are consistent with $\mathcal{C}$ and if infinitely many closed terms over $\mathcal{V}$ are unrestricted in $\bigwedge T$, then T has a model in $\mathcal{C}$.*

Proof. See appendix. ■

Note that the second clause of the previous proposition, the compactness result, applied for instance to $\mathcal{V}$ contain some nonnullary function symbol, and T not containing any occurrence of that symbol. More generally, the compactness result holds for any subset of $\mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$ that are consistent with an arbitrary, independent interpretation of countably many closed terms.

6 Conclusion

We have shown that quantifying over names rather than individuals results in an elegant framework, that does not have to be restricted to semantics based on Henkin or Herbrand structures, thanks to the distinction between the language used to describe a structure, and the language used to talk about a structure. Classical first-order and modal first-order logics can be developed on the basis of this framework, and allow one to avoid the issues raised by combining quantifiers and modalities.

The class of structures involved in the semantics of classical logic is a proper class, not a set. By the Löwenheim–Skolem theorem, this class can be restricted to the class of structures of cardinality at most equal to some ordinal λ, equal to ω if the underlying vocabulary is countable, but that class is still not a set. It is sometimes desirable to deal with sets of structures rather than proper classes. For instance, in [9], a topology is defined over the class of intended interpretations; this requires the class of intended interpretations to be a set. Using the Löwenheim–Skolem theorem and selecting 'representative structures' for all structures of cardinality λ at most, it is possible to indeed select a set of intended interpretations and still be faithful to the semantics defined on the basis of the class of all structures over the underlying vocabulary. The question then arises: how to choose "representative structures"? A natural idea is to use canonical structures, obtained by enriching the vocabulary and making sure that every individual can be identified with the nonempty set of all its names. Then taking the enriched vocabulary as primitive, and considering the original vocabulary as a restriction of the latter, yields the notions that have been presented in this paper. We hope that abstract Kripke frames and quantification over names is not only an effective formal machinery, but can also be supported by epistemic arguments.

Appendix

We start with a couple of technical definitions. They use a member ϵ of $\{0,1\}$, whose purpose is to indicate whether we are interested in the class of all abstract Kripke frames over $\mathcal{V}$ ($\epsilon = 0$) or in the class of all abstract Kripke frames over $\mathcal{V}$ that have constant labeling ($\epsilon = 1$). Let $\mathbb{N}^\star$ denote the set of finite sequences of members of $\mathbb{N}$. We first define a notion of Hintikka set for $(\mathcal{V}, \epsilon)$. Intuitively, such a set $\mathfrak{H}$ will determine, for all members σ of $\mathbb{N}^\star$, an abstract Hintikka frame over $\mathcal{V}$ and a particular abstract structure in this frame, associated with σ. Requiring $\mathfrak{H}$ to contain a pair of the form (σ, ψ) expresses that ψ should be accepted in the abstract Hintikka frame and structure that are determined by $\mathfrak{H}$ and associated with σ.

DEFINITION 17. Let $\epsilon \in \{0,1\}$ be given. A *Hintikka set for* $(\mathcal{V}, \epsilon)$ is any set $\mathfrak{H}$ consisting of pairs of the form (σ, φ), where σ is a member of $\mathbb{N}^\star$ and φ a closed member of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, with the following properties.[2]

[2]Given two finite sequences σ and τ, we let $\sigma \star \tau$ denote the concatenation of σ and τ, and write $\sigma \star t$ in case τ is a singleton (t) that is not itself represented as a sequence.

1. $\mathfrak{H}$ contains no pair of the form $(\sigma, \bigvee \emptyset)$, and $\mathfrak{H}$ does not contain two pairs of the form (σ, ξ) and $(\sigma, \neg\xi)$.

2. If $\mathfrak{H}$ contains a pair of the form $(\sigma, \bigvee X)$ with $X \neq \emptyset$ then $\mathfrak{H}$ contains (σ, ψ) for some $\psi \in X$, and if $\mathfrak{H}$ contains a pair of the form $(\sigma, \bigwedge X)$ then $\mathfrak{H}$ contains (σ, ψ) for all $\psi \in X$.

3. If $\mathfrak{H}$ contains a pair of the form $(\sigma, \exists x\psi)$ then $\mathfrak{H}$ contains $(\sigma, \psi[t/x])$ for some closed term t over $\mathcal{V}$, and if $\mathfrak{H}$ contains a pair of the form $(\sigma, \forall x\psi)$ then $\mathfrak{H}$ contains $(\sigma, \psi[t/x])$ for all closed terms t over $\mathcal{V}$.

4. If $\mathfrak{H}$ contains a pair of the form $(\sigma, \Diamond\psi)$ then $\mathfrak{H}$ contains $(\sigma \star i, \psi)$ for some $i \in \mathbb{N}$, and if $\mathfrak{H}$ contains a pair of the form $(\sigma, \Box\psi)$ then $\mathfrak{H}$ contains $(\sigma \star i, \psi)$ for all $i \in \mathbb{N}$. $(\sigma, t_1 = t_2) \in \mathfrak{H}$ then $(\sigma, t_1 = t_2)$ $(\sigma, t_1 = t_3) \in \mathfrak{H}$. $(\sigma, p(t'_1, \ldots, t'_n)) \in \mathfrak{H}$;

5. If $\mathcal{V}$ contains equality then clauses that deal with it are added (details omitted).

6. Suppose that $\epsilon = 1$. Then for all $\sigma, \tau \in \mathbb{N}^\star$ and for all closed equalities or inequalities ψ over $\mathcal{V}$, $(\sigma, \psi) \in \mathfrak{H}$ iff $(\tau, \psi) \in \mathfrak{H}$.

Let $\epsilon \in \{0, 1\}$ and a Hintikka set $\mathfrak{H}$ for $(\mathcal{V}, \epsilon)$ be given. Then for all members σ of $\mathbb{N}^\star$, there exists a unique abstract structure over $\mathcal{V}$ and a unique abstract Kripke frame over $\mathcal{V}$ that are determined by $\mathfrak{H}$ and associated with σ, as formalized in the next definition.

DEFINITION 18. Let $\epsilon \in \{0, 1\}$ and a Hintikka set $\mathfrak{H}$ for $(\mathcal{V}, \epsilon)$ be given. For all $\sigma \in \mathbb{N}^\star$, let $\mathfrak{M}^\sigma_\mathfrak{H}$ be the (unique) abstract structure over $\mathcal{V}$ with the following properties.

- Two distinct closed terms t_1 and t_2 over $\mathcal{V}$ belong to the same labeling class of $\mathfrak{M}^\sigma_\mathfrak{H}$ iff $\mathcal{V}$ contains equality and $(\sigma, t_1 = t_2) \in \mathfrak{H}$.
- Let $n \in \mathbb{N}$, an n-ary predicate symbol p in $\mathcal{V}$ distinct from equality, and closed terms $t_1, \ldots, t_n$ over $\mathcal{V}$ be given. Then $p(t_1, \ldots, t_n)$ belongs to $\mathfrak{M}^\sigma_\mathfrak{H}(1)$ iff $(\sigma, p(t_1, \ldots, t_n))$ belongs to $\mathfrak{H}$.

We define for all $\sigma \in \mathbb{N}^\star$ an abstract Kripke frame $\mathcal{K}_\mathfrak{H} = (K^\sigma_\mathfrak{H}, R^\sigma_\mathfrak{H})$ by:

- $K^\sigma_\mathfrak{H} = \{\mathfrak{M}^\tau_\mathfrak{H} \mid \tau \in \mathbb{N}^\star,\ \sigma \subseteq \tau\}$;
- for all $\tau_1, \tau_2 \in \mathbb{N}^\star$ that extend σ, $R^\sigma_\mathfrak{H}(\mathfrak{M}^{\tau_1}_\mathfrak{H}, \mathfrak{M}^{\tau_2}_\mathfrak{H})$ holds iff $\tau_1 \subseteq \tau_2$.

The following property directly follows from Definitions 17 and 18:

PROPERTY 19. Let $\epsilon \in \{0, 1\}$ and a Hintikka set $\mathfrak{H}$ for $(\mathcal{V}, \epsilon)$ be given. For all $\sigma \in \mathbb{N}^\star$, the following holds.

1. $K^\sigma_\mathfrak{H}$ is countable.
2. Suppose that $\epsilon = 1$. Then for all $\tau \in \mathbb{N}^\star$, $\mathfrak{M}^\sigma_\mathfrak{H}(0) = \mathfrak{M}^\tau_\mathfrak{H}(0)$.

3. For all closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, if $(\sigma, \varphi) \in \mathfrak{H}$ then $(\mathcal{K}_{\mathfrak{H}}, \mathfrak{M}^{\sigma}_{\mathfrak{H}}) \Vdash \varphi$.

To apply Property 19, it suffices to set ϵ to 0 or 1, choose a closed member φ of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, and exhibit a Hintikka set $\mathfrak{H}$ for $(\mathcal{V}, \epsilon)$ that contains $((), \varphi)$. The set $\mathfrak{H}$ is obtained by constructing a tableau $\mathcal{T}$ for $(\mathcal{V}, \varphi, \epsilon)$, as defined in Definition 21 with the help of the technical notions introduced in Definition 20. The fundamental relationship between $\mathcal{T}$ and $\mathfrak{H}$ is then expressed in Property 22.

DEFINITION 20. Let $\epsilon \in \{0, 1\}$, $\sigma \in \mathbb{N}^\star$, and a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ be given. Let ρ be a finite sequence of pairs of the form (τ, ψ), where τ is a member of $\mathbb{N}^\star$ and ψ a closed member of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$. We say that (σ, φ) is an F-sprout of ρ iff one of the following holds.

1. ρ contains two pairs of the form (σ, ξ) and $(\sigma, \neg\xi)$, and φ is $\bigvee \emptyset$.
2. ρ contains a pair of the form $(\sigma, \bigwedge X)$ with $\varphi \in X$.
3. ρ contains a pair of the form $(\sigma, \forall x \xi)$ and φ is of the form $\xi[t/x]$ for some closed term t over $\mathcal{V}$.
4. ρ contains a pair of the form $(\tau, \Box\varphi)$ and σ is of the form $\tau \star i$.
5. $\epsilon = 1$, φ is an equality or an inequality, and there exists $\tau \in \mathbb{N}^\star$ such that ρ contains an occurrence of (τ, φ).
6. If $\mathcal{V}$ contains equality then extra cases that deal with it are added (details omitted). and ρ contains occurrences of both $(\sigma, t_1 = t_2)$ and occurrences of

DEFINITION 21. Let $\epsilon \in \{0, 1\}$ and a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ be given. We call *tableau for* $(\mathcal{V}, \varphi, \epsilon)$ any tree $\mathcal{T}$ over the set of pairs of the form (σ, ψ), where σ is a member of $\mathbb{N}^\star$ and ψ a closed member of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, with the following properties.

- $((), \varphi)$ is the only child of $()$ in $\mathcal{T}$.
- For all $\rho \in \mathcal{T}$, if the last member of ρ is of the form $(\sigma, \bigvee X)$ with $X \neq \emptyset$, then $\{\rho \star ((\sigma, \psi)) \mid \psi \in X\}$ is the set of children of ρ in $\mathcal{T}$.
- Let $\rho \in \mathcal{T}$ be such that the last member of ρ is of the form $(\sigma, \exists x \psi)$. Let X be the set of closed terms over $\mathcal{V}$ that are unrestricted in all the modal formulas that occur in ρ.
  - If X is finite then $\{\rho \star ((\sigma, \psi[t/x])) \mid t \text{ is a closed term } t \text{ over } \mathcal{V}\}$ is the set of children of ρ in $\mathcal{T}$.
  - If X is infinite then ρ has a unique child which is of the form $\rho \star ((\sigma, \psi[t/x]))$ for some $t \in X$.
- For all $\rho \in \mathcal{T}$, if the last member of ρ is of the form $(\sigma, \bigvee \emptyset)$ then ρ has no child in $\mathcal{T}$.

- For all $\rho \in \mathcal{T}$, if the last member of ρ is of the form $(\sigma, \Diamond\psi)$, then ρ has a unique child which is of the form $\rho \star ((\sigma \star 2i+1, \psi))$ for some $i \in \mathbb{N}$ such that $2i+1$ does not occur in ρ.
- For all nonempty $\rho \in \mathcal{T}$, if the last member of ρ is not of the form $(\sigma, \bigvee X)$, $(\sigma, \exists x\psi)$ or $(\sigma, \Diamond\psi)$, then ρ has a unique child in $\mathcal{T}$, which is either $((), \varphi)$ or one of ρ's F-sprouts.
- For all infinite branches B in $\mathcal{T}$ and for all strict initial segments ρ of B, all F-sprouts of ρ occur in B.

Note that for all branches B in $\mathcal{T}$, either B is infinite and contains no occurrence of the form $(\sigma, \bigvee\emptyset)$, or B is of the form $\rho \star ((\sigma, \bigvee\emptyset))$.

PROPERTY 22. Let $\epsilon \in \{0, 1\}$, a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, and a tableau for $(\mathcal{V}, \varphi, \epsilon)$ be given. If B is an infinite branch in $\mathcal{T}$ then the set of members of B is a Hintikka set for $(\mathcal{V}, \epsilon)$.

Let $\epsilon \in \{0, 1\}$, a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$, and a tableau $\mathcal{T}$ for $(\mathcal{V}, \varphi, \epsilon)$ be given. Then either $\mathcal{T}$ has an infinite branch, or $\mathcal{T}$ is infinite but has no infinite branch, or $\mathcal{T}$ is finite. The first case has been exploited in Property 22. The next proposition considers the third case, which is the key to proving compactness properties.

PROPOSITION 23. *Suppose that $\mathcal{V}$ is countable. Let $\epsilon \in \{0, 1\}$ and a set T of closed members of $\mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$ be such that:*

- *infinitely many closed terms over $\mathcal{V}$ are unrestricted in $\bigwedge T$;*
- *some tableau for $(\mathcal{V}, \bigwedge T, F)$ has no infinite branch.*

Then there exists a finite $D \subseteq T$ and a finite tableau for $(\mathcal{V}, \bigwedge D, F)$.

Proof. Let $\mathcal{T}$ be a tableau for $(\mathcal{V}, \bigwedge T, \epsilon)$ with no infinite branch. Since all members of T are finite, all disjunctions in $\bigwedge T$ are finite. Since infinitely many closed terms over $\mathcal{V}$ are unrestricted in $\bigwedge T$, all member of $\mathcal{T}$ that end in pair of the form $(\sigma, \exists x\psi)$ have a unique child. Hence $\mathcal{T}$ is finitely branching. It follows from König's lemma that $\mathcal{T}$ is finite. Let D denote the (finite) set of members ψ of T such that (σ, ψ) occurs in $\mathcal{T}$ for some $\sigma \in \mathbb{N}^\star$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by replacing all occurrences of $\bigwedge T$ in $\mathcal{T}$ by $\bigwedge D$. It is immediately verified that $\mathcal{T}'$ is a tableau for $(\mathcal{V}, \bigwedge D, \epsilon)$. ■

The next lemma follows immediately from clause 2. in Property 19.

LEMMA 24. *For all Hintikka sets $\mathfrak{H}$ for $(\mathcal{V}, 1)$, $K^{()}_{\mathfrak{H}}$ has constant labeling.*

Putting together the previous results, we can now derive Proposition 16. Let $\mathcal{C}$ be either the set of all abstract Kripke frames over $\mathcal{V}$, or the set of all abstract Kripke frames over $\mathcal{V}$ that have constant labeling. Let a closed $\varphi \in \mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ be given. Let A be the (countable) set of members of $\mathcal{V}$ that occur in φ, and let S be the set of function symbols in $\mathcal{V}$ that do not occur in φ. If S is countable then put $\mathcal{V}^\star = A \cup S$. Otherwise let $\mathcal{V}^\star$ be the union of A with a countably infinite subset of S. Clearly, if infinitely many closed terms over $\mathcal{V}$ are unrestricted in φ then infinitely many closed terms over $\mathcal{V}^\star$ are unrestricted in φ. Put $\epsilon = 0$ if $\mathcal{C}$ is the set of all abstract Kripke frames over $\mathcal{V}$, and $\epsilon = 1$ otherwise. It follows immediately from Lemma 15 that if φ is satisfiable in $\mathcal{C}$ then there exists a tableau for $(\mathcal{V}^\star, \varphi, \epsilon)$ with an infinite branch. The first part of the Proposition 16 follows from the first and third clauses in Property 19, Property 22, and Lemma 24 by taking for φ a closed member of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ that is satisfiable in $\mathcal{C}$. The second part of Proposition 16 follows from Property 22 and Proposition 23 by taking for φ a closed member of $\mathcal{L}^{\mathrm{mod}}_{\omega_1\omega}(\mathcal{V})$ of the form $\bigwedge T$ for some $T \subseteq \mathcal{L}^{\mathrm{mod}}_{\omega\omega}(\mathcal{V})$ that has no model in $\mathcal{C}$.

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