Recap from last time

(I)

(A)
- Convex sets (lines, cones, etc.)
- Menger-convexity in metric spaces.
- 1-span problem.

Let (X, d) be a metric space.

depends on 'd'

(B)
- **Complete** if all Cauchy sequences in X converge to a point in X. ~~($\mathbb{Q}$~~ So towns in NOT complete with $\|\cdot\|_2$)
- **Locally compact**: If every point of X has a compact nbhd (i.e., for us, we can put an ϵ-Ball around each point in X)
- $\mathbb{Q}$ is not l.c. (around no point can we put a ball)
-  $\{(0,0)\} \cup \{(x,0) \mid x > 0\} \subset \mathbb{R}^2$

  $\{(0,0)\}$ does NOT have a compact nhbd.

Fundamental Theorem of Menger convexity:

Thm: Suppose (X, d) is complete and locally compact. Then,

(D)

(X, d) is meng. cvx $\Longleftrightarrow$

$\forall x, y \in X, \exists m \in X$ st

$d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$ (midpoint cvx)

$\Longleftrightarrow$ Any two $x, y \in X$ are joined by a geodesic

First this!

Defn: A geodesic is a continuous path of shortest length between two points in (X, d)!

(C)

Let $x, y \in X$, $t \in [0,1]$

$\gamma(t)$ is such that $\gamma(0) = x$, $\gamma(1) = y$ and

$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| \, d(x, y)$. $\forall t_1, t_2 \in (0,1)$

Example: $(\mathbb{R}^n, \|\cdot\|_2)$ is a geodesic metric space; here m-cvx ~~is not~~ just ordinary convexity.

Question: Is $(\mathbb{R}_{++}, |\log x - \log y|)$ complete?

Yes: each point in $\mathbb{R}_{++}$ can be written as e^{u_i}, $u_i \in \mathbb{R}$ and thus $|\log x - \log y| = |u_i - v_i|$; But $(\mathbb{R}, |\cdot|)$ is complete.

CONVEX FUNCTIONS

- Midpoint convex

  Let $x, y \in \mathcal{X}$ then f is called midpoint cvx if

  $$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \quad \forall x,y \in \mathcal{X}$$

- Jensen-convex, hereafter just convex.

  $$f(\underline{(1-\alpha)x + \alpha y}) \le (1-\alpha)f(x) + \alpha f(y) \quad \forall \alpha \in [0,1].$$

Clearly dom f better be convex.

Already mention here cvx on $(\mathcal{X}, d)$, before going onto the next theorem.

Theorem: (Jensen, 1905): Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then 'f' is cvx iff it is midpt cvx.

Proof: (Invoke our general result); clearly only sufficiency part needs any proof. Argue by contradiction.

Suppose $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$ $\forall x,y \in I$,

But that $f((1-\alpha)x + \alpha y) \le$ violated. (*)

III

– then, consider the continuous function

$$g(\alpha) := f((1-\alpha)x + \alpha y) - (1-\alpha)f(x) - \alpha f(y).$$

- Since (**) is violated, $\Rightarrow$ max value of 'g' on $[0,1]$ will be ~~≥ 0~~ some num $M > 0$.
- let α_0 be the smallest $\alpha \in (0,1)$ for which $g(\alpha) = M$.
- Let $\delta > 0$ be small enough so that (why is $\alpha_0 \in (0,1)$ not on bdry?)

$$[\alpha_0 - \delta, \alpha_0 + \delta] \subset [0,1].$$

- let
$$\bar{x} := (1 - \alpha_0 - \delta)x + (\alpha_0 + \delta)y$$
$$\bar{y} := (1 - \alpha_0 + \delta)x + (\alpha_0 - \delta)y$$
(verify these lie in the interval)

- ~~$g(\alpha_0)$~~

- $$f\left(\frac{\bar{x}+\bar{y}}{2}\right) = f\left(\frac{(1-\alpha_0)x + \alpha_0 y}{\cancel{\ }}\right)$$
$$\leq \frac{f((1-\alpha_0)x) + f(\alpha_0 y)}{2}$$

- $$g(\alpha_0) \leq \frac{g(\alpha_0 - \delta) + g(\alpha_0 + \delta)}{2} < M,$$

a contradiction. $\square$

~~$g(\alpha_0) = f((1-\alpha_0)x + \alpha_0 y) - (1-\alpha_0)f(x) - \alpha_0 f(y)$~~

$g(\alpha_0 - \delta) \leftarrow$

(Cite: Krasnoselskii for proof !)

~~# Convex functions in $(\mathcal{X}, d)$ for a geodesic metric space.~~

~~Let $\gamma(\alpha) := (1-\alpha)x \oplus \alpha y$ denote a geodesic between $x, y \in \mathcal{X}$.~~

~~$f: \mathcal{X} \to \mathbb{R}$ is called geodesically convex~~

~~if $f((1-\alpha)x \oplus \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$~~

We will return to such geodesically convex functions later when studying "geometric optimization".

Some ways to recognize convex functions:

— Contin. & midpoint convex

— If f is diff it is convex iff

$$f(x) \geq f(y) + \langle \nabla f(y), x-y \rangle \quad \forall x,y \in \operatorname{dom} f$$

— If f is twice diff, convex iff

$$\nabla^2 f(x) \succeq 0 \quad \forall x \in \operatorname{dom} f.$$

— $f: I \to \mathbb{R}$ cvx iff its derivative f' is $\uparrow$.

— $f: \mathbb{R}^n \to \mathbb{R}$ cvx iff " " is a monotone operator

i.e. $\langle \nabla f(x) - \nabla f(y), x-y \rangle \geq 0 \quad \forall x,y \in \cdots$

~~— $f: I \to \mathbb{R}$ cvx iff it takes its maxima (for all subintervals f takes its maxima at endpoints)~~

— We'll see other methods later on.

— Restriction to line: $g(t) = f(x + ty)$ for arbit $x, y \in \mathcal{X}$, $t \in \mathbb{R}$ (or whatever is suitable).

We'll allow f to take on $\pm\infty$ as a value.

This allows us to just write $f: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$.

Important conventions ~~$-\infty < x \leq \infty \ \forall x \in \mathbb{R}$~~

(1) ~~$x + \infty = \infty + x = \infty$~~ th.

(2) ~~$-\infty < x \leq \infty$~~ ;

- $x + \infty = \infty + x$ for $-\infty < x \leq \infty$
- $x - \infty = -\infty + x$ for $-\infty \leq x < \infty$
- $x\infty = \infty x = \infty$ for $0 < x \leq \infty$
- $0 \cdot \infty = \infty \cdot 0 = 0$
- $-(-\infty) = +\infty$
- $\inf \phi = +\infty$; $\sup \phi = -\infty$.
- Avoid $\infty - \infty$, $-\infty + \infty$: akin to div. by zero.

Proper cvx fn: $f > -\infty$ everywhere and $< +\infty$ at at atleast one point.

($\exists$ $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\} \neq \phi$ and $f|_{\operatorname{dom} f}$ is finite)

With this : Ee.

$$f(x) = \begin{cases} 1/x & x > 0 \\ +\infty & x \leq 0 \end{cases}$$

$$f(x) = \begin{cases} -\log x, & x > 0 \\ +\infty & x \leq 0 \end{cases}$$

are convex

Some Important convex functions

ID

— Examples:

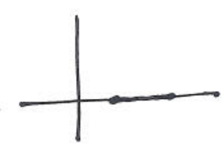
(a) Read chapter 3 of BV.

(b) Indicator function: Let $C \subset \mathbb{R}^n$ be nonempty.

$$\delta_C(x) := \begin{cases} 0, & x \in C \\ +\infty, & x \notin C. \end{cases}$$

Verify: δ_C is closed and convex iff C is closed and convex.

By closed we mean: $\operatorname{epi} f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le t\}$ is a closed set.



(b) Support function.

Let C be a nonempty set.

$$\sigma_C(x) := \sup_{z \in C} z^T x$$

(c) More generally, let ~~$f(x,y)$ be~~ $f_\alpha(x)$ be $f(x,y)$:

a family of functions indexed by $y \in \mathcal{Y}$ such that $f(x,y)$ is convex in x for each y. Then

$$f(x) := \sup_{y \in \mathcal{Y}} f(x,y) \text{ is cvx.}$$

(Note: for world of cvx., akin to "marginalizing out")

(d) Consequence: Fenchel-conjugate:

Let $f(x) : \mathbb{R}^n \to \overline{\mathbb{R}}$ be any function. Then

$$f^*(z) := \sup_{x \in \mathbb{R}^n} \langle z, x \rangle - f(x) \text{ is cvx.}$$

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For: **cangiuli**

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(e) Let $\mathcal{Y}$ be a nonempty convex set. Let $h(x,y)$ be jointly convex, i.e. cvx in (x,y).

Then $f(x) := \inf_{y \in \mathcal{Y}} h(x,y)$ is a convex fn of x. (provided $f(x) > -\infty$).

Proof: lecture notes

$$\cancel{\inf_y h((1-\lambda)u+\lambda v, y) \leq h((1-\lambda)u+\lambda v, (1-\lambda)y_1+\lambda y_2) \leq (1-\lambda)h(u,y_1)+\lambda h(v,y_2) \leq}$$

Example: $d_C(x) := \inf\{\|y-x\| \mid y \in C\}$ is cvx!

VII

— Read Chap 3 of BV.

~~— Piecewise affine:~~

— Norms on $\mathbb{R}^n$:

Let $f: \mathbb{R}^n \to \mathbb{R}_+$ be a function that satisfies

- $f(x) \geq 0 \;\; \forall x \in \mathbb{R}^n$
  $f(x) = 0$ iff $x = 0$ (+ve def.)
- $f(\lambda x) = |\lambda| f(x)$: +ve homogenity
- $f(x+y) \leq f(x) + f(y)$: Subadditivity

Then f is called a Norm.

We denote such an f by $\| \cdot \|$.

Thm: All norms are convex.

Examples: l_p norms: $\|x\|_p := \left[\sum_{i=1} |x_i|^p\right]^{1/p}$ $\quad 1 \leq p < \infty$

l_∞ norm: $\|x\|_\infty := \max\limits_{1 \leq i \leq n} |x_i|$.

Exercise: Verify that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

Mixed-norm: $l_{p,q}$-norm: Let $x \in \mathbb{R}^{n_1+n_2+\dots+n_k}$ be partitioned into subvectors $(x_1, \dots, x_k)$.

Let $1 \leq p, q \leq \infty$.

$l_{p,q}$-norm $\quad \|x\|_{p,q} := \left(\sum_{i=1}^k \|x_i\|_q^p\right)^{1/p}$.

(Shows up freq. in ML.)

(VIII).

— Matrix norms

- **Frobenius norm**: $A \in \mathbb{C}^{n \times m}$.

$$\|A\|_F := \sqrt{\mathrm{tr}(A^* A)}$$

Well-posed because: $\|A\|_F := \sqrt{\sum_{ij} |a_{ij}|^2}$.

- **Induced norm**: Let $A \in \mathbb{R}^{m \times n}$; $\|\cdot\|_v$ any vector norm.

$$\|A\|_v := \sup_{\|x\| \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

Eg: $\|A\|_1$:= matrix 1-norm
$\|A\|_\infty$: " ∞-norm.

| $\|A\|_p$: NP-Hard to compute |

- **Schatten p-norm**: $\|A\|_{(p)}$ or $\|A\|_p$ (don't confuse with above)

$$:= \|\sigma(A)\|_p \qquad \sigma(A) \text{ sing values}$$

- Ky-Fan norm (cf. for vectors

$$\|x\|_{(k)} := \sum_{i=1}^{k} |x_i|^{\downarrow} \quad (\text{Sorted in decr abs.})$$

$$\|A\|_{(k)} := \sum_{i=1}^{k} \sigma_i(A).$$

- **Operator 2-norm** say $A \in \mathbb{C}^{m \times n}$;

$$\|A\|_2 := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

Claim $\|A\|_2 := \sigma_1(A)$ (Prove it!!)

~~Notes: $\|A\|_p$~~

- Dual- norms

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$.

Its dual norm is

$$\|u\|_* := \sup \{ u^T x \mid \|x\| \le 1 \}.$$

(Notice this is the support function of the unit norm ball)

- Minkowski gauge functional.

(Norms are ~~awesome~~ distinguished because their epi is a cone)

- Let C be any nonempty convex set.

Gauge: $k(x) := \inf \{ \mu \geq 0 \mid x \in \mu C \}$

Verify: k is nonneg; $k(\alpha x) = \alpha k(x)$, $\alpha \geq 0$.
$k(0) = 0$.

~~• Is k subadditive? superadd? $k(x+y) = \inf \{ \mu \geq 0 \mid x+y \in \mu C \}$~~

Notice if $x \notin \mu C$ for any $\mu \geq 0$, by our convention
$k(x) = +\infty$.

So gauges can be infinite even though $x \in \mathbb{R}^n$ is NOT.

If gauge is finite, symmetric, and >0 (except at 0), we call it a norm.

In the paper "Convex geometry of linear inverse problems", V. Chandrasekaran et. al. introduced "Atomic norms" by writing them as gauges of suitable convex hulls.

To close:

Prop: Let $\|\cdot\| \in \mathbb{R}^n$ be any norm. Then
$$f^*(z) := \delta_{\|\cdot\| \leq 1}(z)$$
~~as soon as conjugate and norm~~

~~Stay it~~ ~~easy~~ on.

Proof: Don't mix $f^*(z)$ with $\|\cdot\|_*$ the dual-norm;

Consider two cases: $\|z\|_* > 1$ and $\|z\|_* \leq 1$.

(i) $\|z\|_* = \sup_{u:} \{u^T x \mid \|u\| \leq 1\}$ (Compact set $\|x\| \leq 1$ so sup is attained)

$\Rightarrow \exists\, u$ st $\|u\| \leq 1$ and $u^T x > 1$.

$f^*(z) := \sup\, u^T x - f($

(XI)?

<u>Claim</u>: Let $f(x) = \|x\|$; Then $f^*(z) := \delta_{\|\cdot\| \le 1}(z)$.

<u>Proof</u>: Consider two cases: (i) $\|z\|_* > 1$ and (ii) $\|z\|_* \le 1$.

(i) $\|z\|_* = \sup \{ u^T z \mid \|u\| \le 1 \} \Rightarrow \exists\, u$ st $\|u\| \le 1 \wedge \underline{u^T z > 1}$.

$$f^*(z) = \sup_x \; z^T x - f(x);$$

Select $\underline{x} = \alpha \underline{u}$ and let $\alpha \to \infty$

$$\alpha z^T u - \alpha \|u\| = \alpha (z^T u - \|u\|) \to \infty$$

(since $z^T u > 1$).

(ii) Since $\|z\|_* \le 1$

$$\|z\|_* = \sup_{u \ne 0} \frac{u^T z}{\|u\|} \Rightarrow u^T z \le \|u\| \cdot \|z\|_*$$

[why could we write: <u>Since</u> <u>homog</u>, we may assume $\underline{\|u\| = 1}$; then <u>optimize</u> over this set]

$$\Rightarrow z^T x - \|x\| \le \|x\| (\|z\|_* - 1) \le 0$$

So $\underline{x = 0}$ maximizes this, hence

$f^*(z) = 0$ in this case.

<u>Thus</u>: $$f^*(z) = \begin{cases} 0 & \text{if } \|z\|_* \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$ ☒

<u>Challenge Problem</u>: Let $\underline{x_1, x_2, \dots, x_n > 0}$.

<u>Finish here</u>

$h_n(x)$ $\qquad g(t) := h_n(x + tv)$

$\frac{1}{x+t}$ $\quad \frac{1}{x+tv_1} + \frac{1}{y+tv} - \frac{1}{x+y+t(v_1+v_2)}$

$\sum_i \frac{1}{x_i + tv_i} + \sum_{i<j} (-1) \frac{1}{x_i + x_j + t(v_i + v_j)} \dots$ eh. is convex?

$= \frac{2v}{(x+tv)^3}$ $\quad -\frac{v_1}{(x+tv)^2} - \frac{v}{(y+tv)^2} + \frac{(v_1+v)}{(x+y+t(v))^2}$

(XII) Challenge 2

$$F_\mu(x) := \frac{1}{\Gamma(\mu+1)}\int_0^\infty \frac{t^\mu}{e^{t-x}+1}\,dt$$

Claim: $\left[F_\mu(x)\right]^{\frac{1}{\mu+1}}$

$$-\mathrm{Li}_{\mu+1}(-e^x) = F_\mu(x)$$

Claim $(\quad)^{1/\mu+1} = \left[F_\mu(x)\right]^{1/\mu+1}$ $\quad \mu \geq 0$, $x \in \mathbb{R}$

$F_0 = \log(1+e^x)$ is cvx!

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$\log\det(X)$ is concave on $X \succ 0$

V.O

Proof: we show equivalently that

$$\left|\frac{X+Y}{2}\right| \geq |X|^{1/2}|Y|^{1/2} \qquad (\text{AM-GM ineq.})$$

Divide both sides by $|X|$, this reduces to

$$\left|\frac{I+X^{-1}Y}{2}\right| \geq |X^{-1}Y|^{1/2}$$

Let λ_i be ith eigenvalue of $X^{-1}Y$, since det = prod eigvals

~~$\det(Z) = \prod_i \lambda_i(Z)$~~

$$\Rightarrow \prod_i \left(\frac{1+\lambda_i}{2}\right) \geq \prod_i \sqrt{\lambda_i}$$

which is obviously true

Non-~~trivial~~ elem fact used

$$A, B \succ 0 \Rightarrow \lambda(AB) > 0$$

$$\lambda(AB) = \lambda(B^{-1/2} A B^{1/2})$$

$$\lambda(AB) = \lambda(B^{+1/2} A B \cdot B^{-1/2})$$

$$= \lambda(B^{1/2} A B^{1/2}) > 0.$$

if $f: \mathbb{R}^n \to \mathbb{R}$ is cvx

then $A: \mathbb{R}^m \to \mathbb{R}^n$ is an affine map

then $f \circ A$ is cvx : ultra simple

$f(A(x)) : f(Ax+b) =$ $\quad x \in \mathbb{R}^m$