Constructing the Propositional Truncation using Non-recursive HITs

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In *Homotopy Type Theory* (HoTT) there are *Higher Inductive Types*, generalizing Inductive Types.

Goal: Reduce complicated Higher Inductive Types to simpler ones.

Analogue: In Extensional Type Theory, we can reduce all inductive types to W-types and Σ-types.

Homotopy Type Theory

Homotopy Type Theory combines Type Theory with Homotopy Theory.

	Type Theory	**Logic**
A	Type	Formula
$a : A$	Term	Proof
$A + B$	Sum Type	Disjunction
$A \to B$	Function Type	Implication
$P : A \to \text{Type}$	Dependent Type	Predicate
$\Pi(x : A),\ P(x)$	Dep. Fn. Type	U. Quantifier
$a =_A b$	Identity	Equality

Homotopy Type Theory

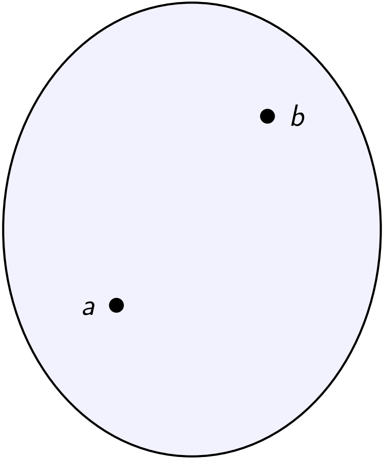
Homotopy Type Theory combines Type Theory with Homotopy Theory.

	Type Theory	**Logic**	**Homotopy Theory**
A	Type	Formula	Space*
$a : A$	Term	Proof	Point
$A + B$	Sum Type	Disjunction	Coproduct of spaces
$A \to B$	Function Type	Implication	Mapping space
$P : A \to \text{Type}$	Dependent Type	Predicate	Fibration
$\Pi(x : A),\ P(x)$	Dep. Fn. Type	U. Quantifier	Dep. product space
$a =_A b$	Identity	Equality	Path space

Types as spaces

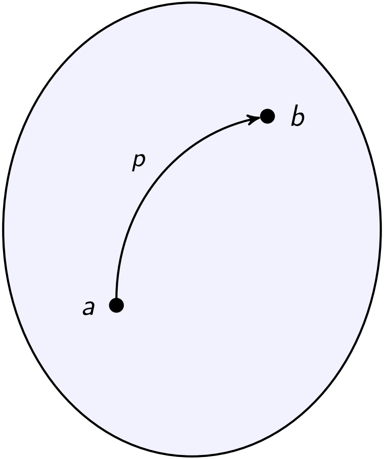
A type A can have

- points $a, b : A$



A type A can have

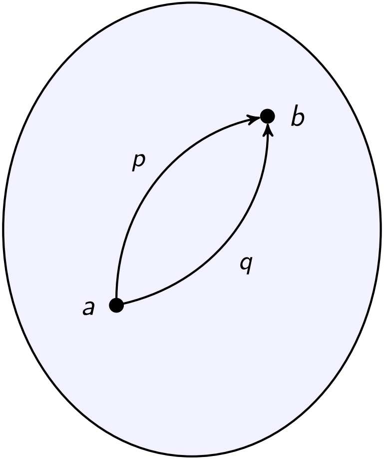
- paths $p : a =_A b$
- points $a, b : A$



Types as spaces

A type A can have

- paths $q, p : a =_A b$
- points $a, b : A$

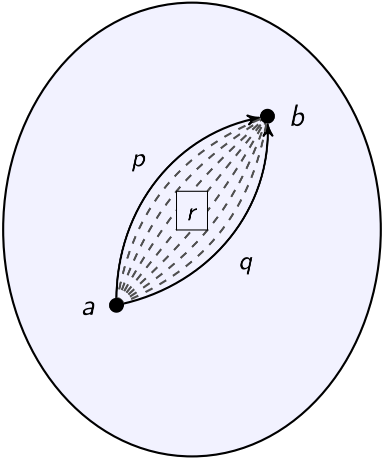


Types as spaces

A type A can have

$$\vdots$$

- paths between paths $r : p = q$
- paths $q, p : a =_A b$
- points $a, b : A$



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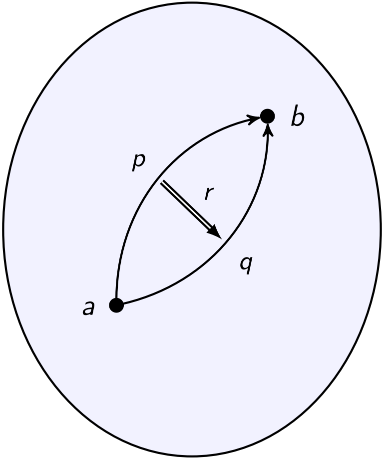
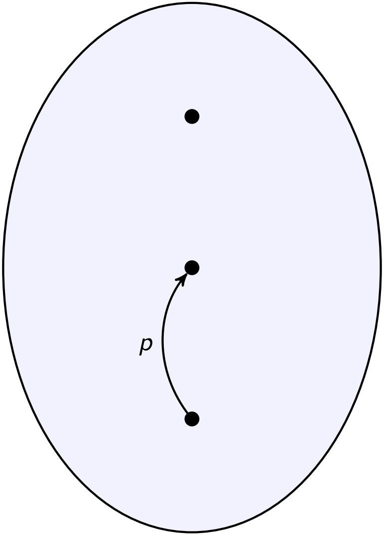


Operations on paths



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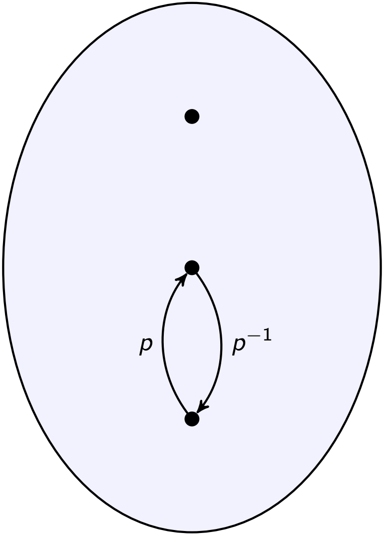
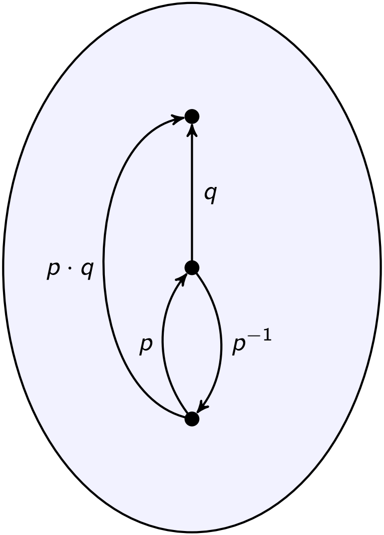


Operations on paths

We can

- invert paths (symmetry);
- concatenate paths (transitivity);

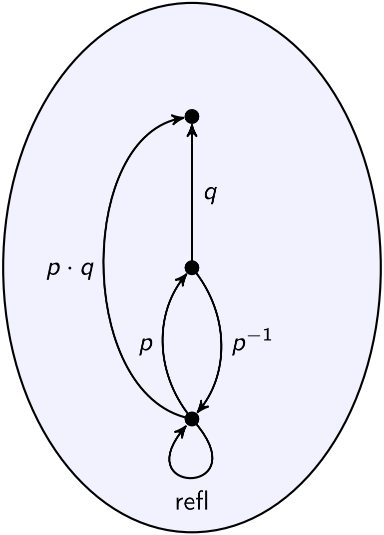


Operations on paths

We can

- invert paths (symmetry);
- concatenate paths (transitivity);
- make identity paths (reflexivity).

  (and more)



Truncated Types

Some types have trivial higher structure.

Type

$\vdots$

- paths between paths $r, s : p = q$
- paths $q, p : a =_A b$
- points $a, b : A$

Truncated Types

Some types have trivial higher structure.

Set	**Type**
$\vdots$	$\vdots$
[at most one path between paths]	paths between paths $r, s : p = q$
[at most one path between points]	paths $q, p : a =_A b$
points $a, b : A$	points $a, b : A$

Truncated Types

Some types have trivial higher structure.

Mere Proposition

$\vdots$

- [at most one path between paths]
- [at most one path between points]
- [at most one point]

Set

$\vdots$

- [at most one path between paths]
- [at most one path between points]
- points $a, b : A$

Type

$\vdots$

- paths between paths $r, s : p = q$
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Propositional Truncation

Given A, we can form the *Propositional truncation* $\|A\|$.

$\|A\|$ is the **mere proposition** specifying whether A is inhabited.

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A		$\|A\|$
$\vdots$		$\vdots$
• paths between paths	$\Rightarrow$	• [at most one path between paths]
• paths		• [at most one path between points]
• points		• [at most one point]

Propositional Truncation (usage)

An element of $A + B$ specifies whether A is inhabited or B is inhabited.

Using the propositional truncation we can define a proof irrelevant disjunction which does not reveal its witness.

$$A \vee B :\equiv \|A + B\|$$

$$\exists x, P(x) :\equiv \|\Sigma x,\ P(x)\|$$

Higher Inductive Types

In HoTT we use *Higher Inductive Types* to construct types with nontrivial paths.

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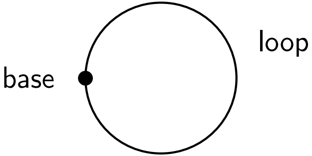
Example: the circle

`HIT` $S^1 :=$

- base : S^1
- loop : base $=$ base



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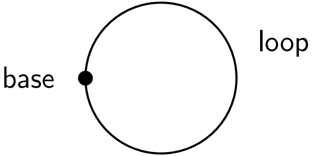

In the circle we have loop $\neq$ refl. Hence we have paths

$$\cdots \quad \text{loop}^{-1}, \quad \text{refl}, \quad \text{loop}, \quad \text{loop} \cdot \text{loop}, \quad \text{loop} \cdot \text{loop} \cdot \text{loop}, \quad \cdots$$

which are all different.

Sequential Colimits

Given

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$$

we can define the colimit of this sequence.

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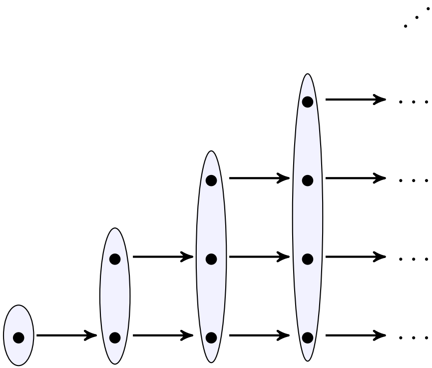

The colimit of this diagram is $\mathbb{N}$.

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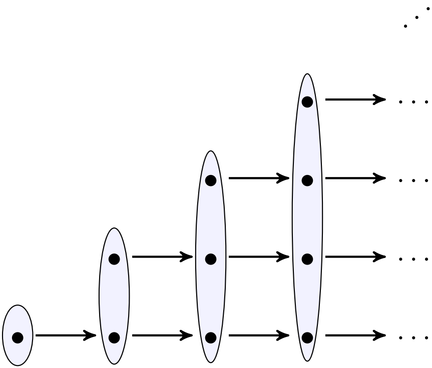

This colimit is a HIT:

`HIT` colimit(A, f) :=

- $i : \Pi(n : \mathbb{N}),\ A_n \to$ colimit(A, f)
- $g : \Pi(n : \mathbb{N}),\ \Pi(a : A_n),$ $i_{n+1}(f_n(a)) = i_n(a)$

The colimit of this diagram is $\mathbb{N}$.

Propositional Truncation as HIT

The Propositional Truncation $\|-\|$ is also a HIT:

`HIT` $\|A\| :=$

- $|-| : A \to \|A\|$
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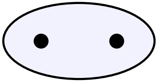
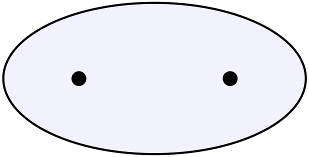


$\|A\|$
(partial structure)



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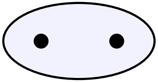

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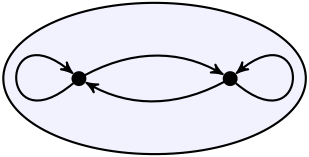

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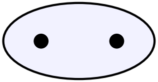

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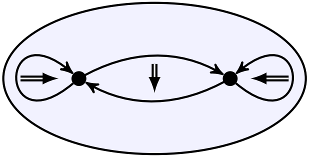

HITs are not well understood

- There is no general theory describing which HITs are allowed.
- Which restrictions are needed for the constructors?
- It will help if we can reduce complicated HITs to basic HITs.
- The definition of the propositional truncation is impredicative. Can we give a predicative construction?

Construction of the Propositional truncation

We will define the Propositional Truncation as a colimit.
At every step we will apply the *one-step truncation*, which is the following HIT.

$\texttt{HIT}\ \{A\} :=$

- $f : A \to \{A\}$
- $e : \Pi(x, y : A),\ f(x) = f(y)$

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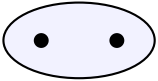

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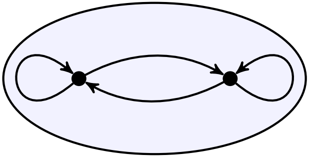

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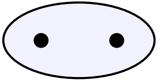

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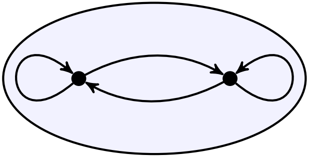

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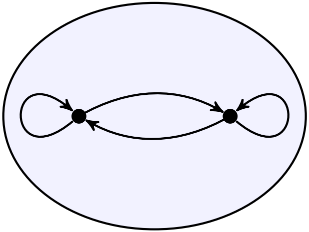

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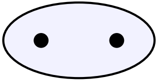

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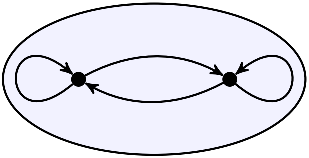

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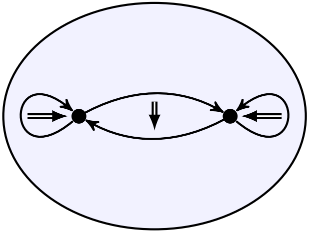

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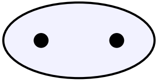

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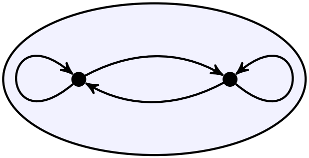

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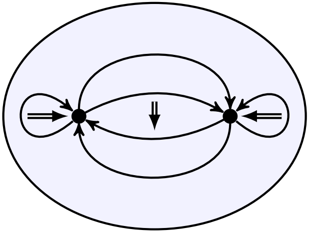

$\vdots$

Construction of the Propositional truncation

We obtain the diagram

$$A \xrightarrow{f} \{A\} \xrightarrow{f} \{\{A\}\} \xrightarrow{f} \{\{\{A\}\}\} \xrightarrow{f} \cdots \tag{1}$$

Theorem

The colimit of diagram (1) *is the propositional truncation* $\|A\|$.

Corollary

A function in $\|A\| \to B$ *is the same as a cocone over* (1), *for any type* B.

Formalization

This result is formally proven in the proof assistant Lean.

In Lean, the HoTT mode is (mostly) just the standard mode, without `Prop`.

There is no proof assistant with good support for HITs.

In Lean we added two HITs as a primitive types, and we can define most of the commonly used HITs in terms of these.

Formalization

```
definition trunc.{u} (A : Type.{u}) : Type.{u}
definition tr (A : Type} : A → trunc A
definition is_hprop_trunc (A : Type) : is_hprop (trunc A)
definition trunc.rec {A : Type} {P : trunc A → Type}
  [Pt : Π(x : trunc A), is_hprop (P x)]
  (H : Π(a : A), P (tr a)) : Π(x : trunc A), P x
```

```
definition elim2_equiv (A P : Type) : (trunc A → P) ≃
  Σ(h : Π{n}, n_step_tr A n → P),
    Π(n : ℕ) (a : n_step_tr A n),
      @h (succ n) (one_step_tr.tr a) = h a
```

Conclusions

- HITs are generally not very well understood.
- We can construct the propositional truncation (a recusive HIT) using simpler (nonrecursive) HITs.
- Conjecture: A large class of HITs can be reduced to a single HIT, the *homotopy coequalizer*.

Thank you

The Lean formalization is available at:

`https://github.com/fpvandoorn/leansnippets/blob/master/cpp.hlean`