Lessons Learned from Formalizing Local Convex Integration

Floris van Doorn

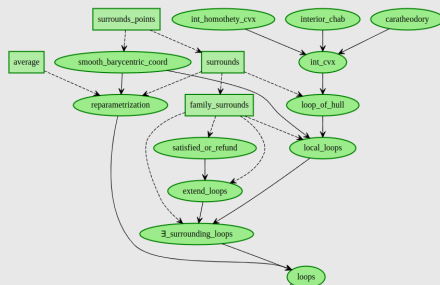University of Paris-Saclay

May 10, 2022

Sphere Eversion Project

This talk is a continuation of Patrick's talk from the morning.
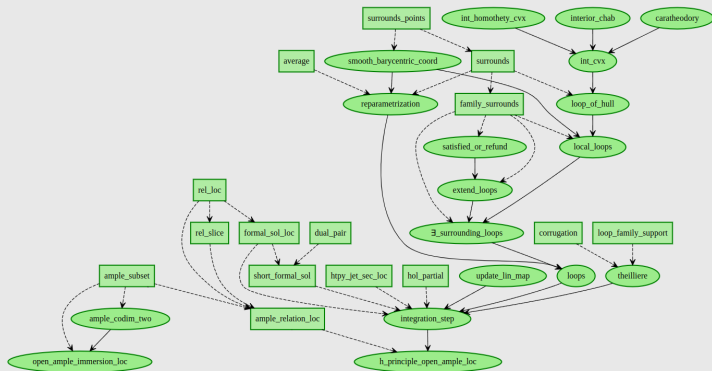
The goal of the Sphere Eversion project is to formalize convex integration, with sphere eversion as immediate application.

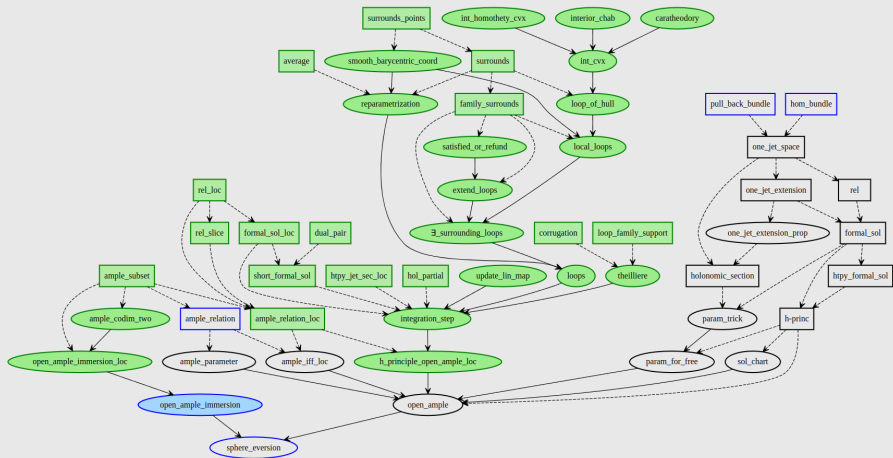This project is a collaboration with Patrick Massot and Oliver Nash.
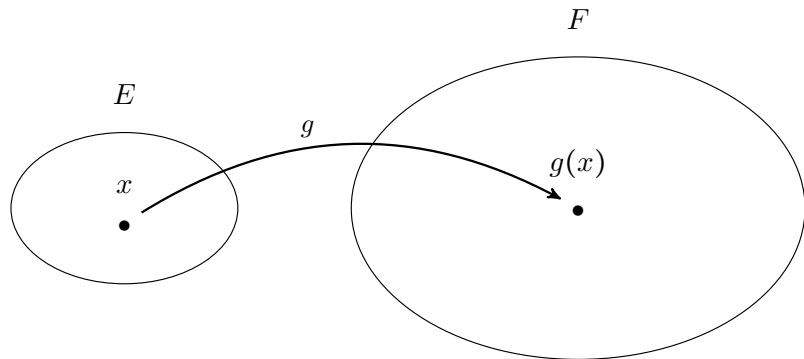
Dependency Graph

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Picture
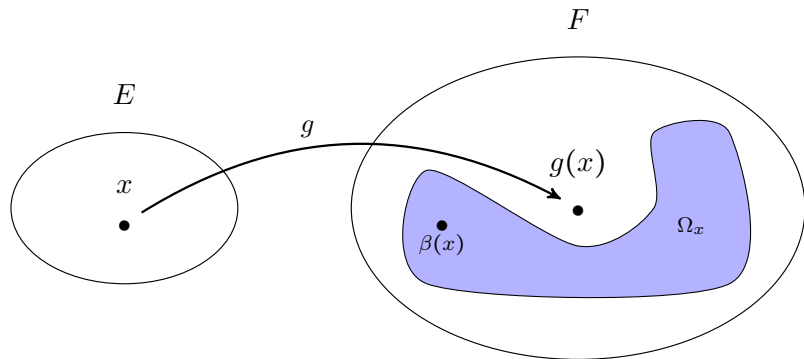
Picture

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Main Proposition

Proposition

Let Ω be an open set in $E \times F$ such that, for each x the set
$\Omega_x := \{y \in F \mid (x, y) \in \Omega\}$ is connected in F.
Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x.
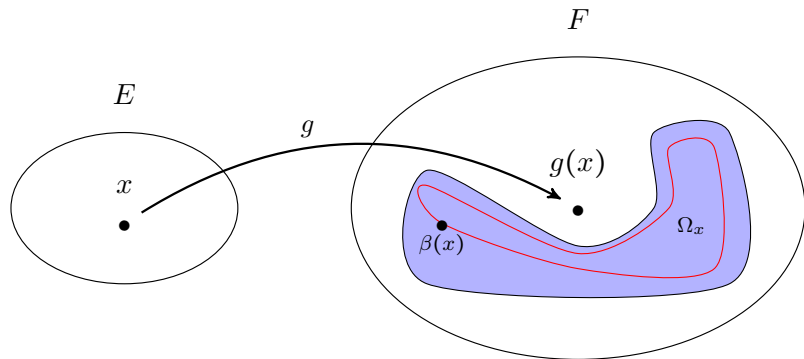*Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.*

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Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.
Then there exists a smooth family of loops

$$\gamma : E \times [0, 1] \times \mathbb{S}^1 \to F, \quad (x, t, s) \mapsto \gamma_x^t(s)$$

such that, for all $x \in E$, all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

- $\gamma_x^t(s) \in \Omega_x$
- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- *the average of γ_x^1 is $\overline{\gamma_x^1} = g(x)$*

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$\Omega_x := \{y \in F \mid (x,y) \in \Omega\}$ is connected in F. Let K a compact set in E.
Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x, and $g(x) = \beta(x)$ near K.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.
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- the average of γ_x^1 is $\overline{\gamma_x^1} = g(x)$
- $\gamma_x^t(s) = \beta(x)$ if x is near K.

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Pointwise version

Proposition

Let Ω be an open connected set in F.

Let $\beta, g \in F$.
Assume that $\beta \in \Omega$.
Suppose that g is in the convex hull of Ω.
Then there exists a smooth family of loops

$$\gamma : [0,1] \times \mathbb{S}^1 \to F, \quad (t,s) \mapsto \gamma^t(s)$$

such that for all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

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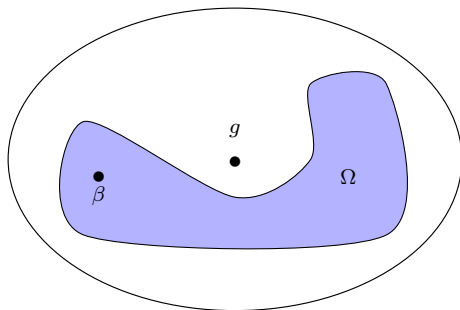- γ^1 *surrounds g.*

Pointwise solution

Definition

- A point $x \in F$ is surrounded by points a finite set of points $\{p_i\}$ if those points form an affine basis and there exist weights $w_i \in (0,1)$ with sum 1 such that $x = \sum_i w_i p_i$.
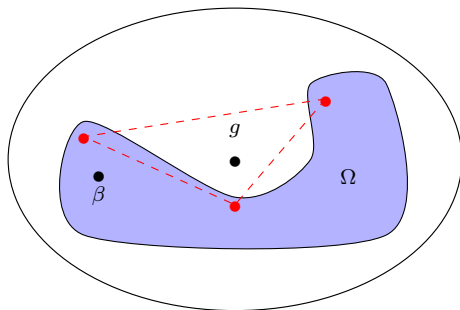- A set A surrounds x if there is a collection of points in A surrounding x.

Lemma

If a point $x \in F$ lies in the convex hull of an open set P then P surrounds x.

Pointwise solution
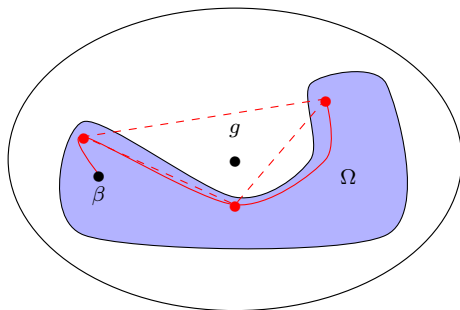
Pointwise solution

Proof.

By the lemma, pick a collection of points surrounding g.

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By the lemma, pick a collection of points surrounding g. Since Ω is open and connected, we can find a path starting at β through these points.

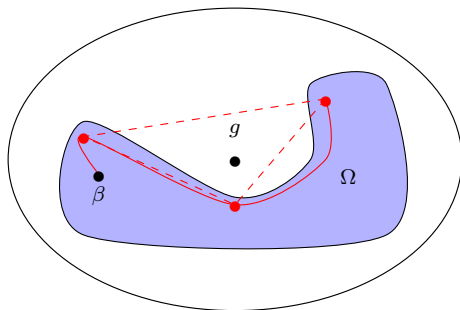$\square$

Pointwise solution

Proof.

By the lemma, pick a collection of points surrounding g. Since Ω is open and connected, we can find a path starting at β through these points. Retracing the same path back to β, we obtain a path that is homotopic to the constant path. $\square$

Local solution

Suppose we have such a γ_{x_0} based at $\beta(x_0)$ surrounding $g(x_0)$.

We can extend this to a local solution around x_0:

$$\gamma_x^t(s) = \gamma_{x_0}^t(s) + \beta(x) - \beta(x_0)$$

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Then:

- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- For x sufficiently close to x_0 we have $\gamma_x^t(s) \in \Omega_x$
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$\forall^f$ x in $\mathcal{N}$ x_0, $\forall$ (t $\in$ I) (s $\in$ I), (x, γ x t s) $\in \Omega$
$\forall^f$ x in $\mathcal{N}$ x_0, (γ x 1).surrounds (g x)

Local solution

We write $\gamma \in \mathcal{L}(U)$ for such a γ defined on $U \subseteq E$.

So $\gamma \in \mathcal{L}(U)$ means that γ is a continuous family of loops

$$\gamma : E \times [0,1] \times \mathbb{S}^1 \to F, \quad (x,t,s) \mapsto \gamma_x^t(s)$$

such that for every $x \in U$, every $t \in [0,1]$ and every $s \in \mathbb{S}^1$,

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Gluing solutions

We want to glue these local solutions together, to obtain a global solution defined on all of E.

Let's start with two local solutions $\gamma_0 \in \mathcal{L}(U_0)$ and $\gamma_1 \in \mathcal{L}(U_1)$.

We want to get a solution on $U_0 \cup U_1$.

One thing we need is to transition between γ_0 to γ_1 on $U_0 \cap U_1$.

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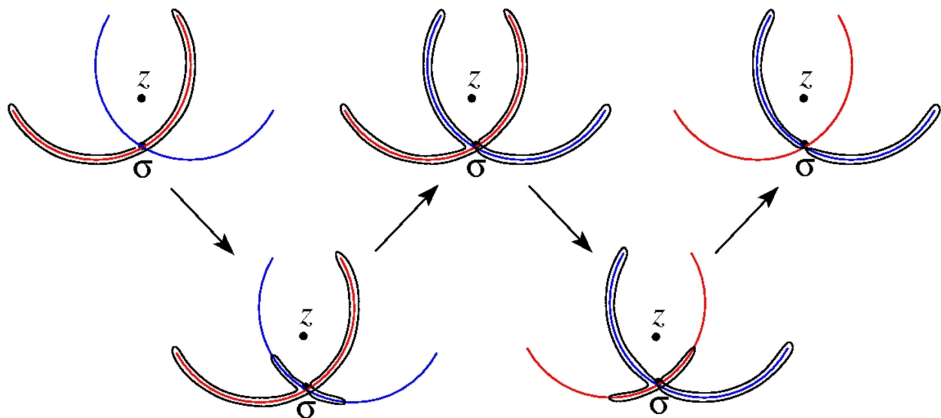
Lemma

If $U \subseteq E$ then $\mathcal{L}(U)$ is contractible: if $\gamma_0, \gamma_1 \in \mathcal{L}(U)$ then there is a continuous homotopy

$$\delta : [0,1] \times E \times [0,1] \times \mathbb{S}^1 \to F, \quad (\tau, x, t, s) \mapsto \delta_{\tau,x}^t(s)$$
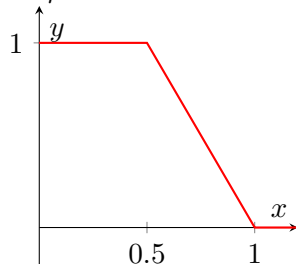
such that $\delta_\tau \in \mathcal{L}(U)$ and $\delta_i = \gamma_i$ for $i \in \{0, 1\}$.

Transitioning between solutions

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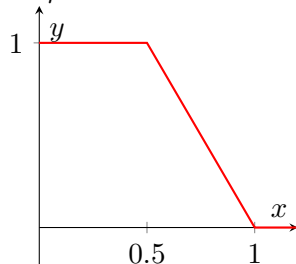Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following piecewise affine function:

We can define the homotopy δ as follows:

- $\delta_{\tau,x}^t$ moves along the loop $\gamma_{0,x}^{\rho(\tau)t}$ once on $[0, 1-\tau]$ (if $\tau < 1$)
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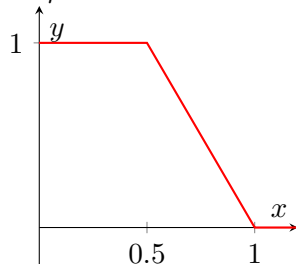Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following piecewise affine function:

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Note that the image of $\delta_{\tau,x}^1$ contains the image of $\gamma_{0,x}^1$ for $\tau \leq \frac{1}{2}$, and the image of $\gamma_{1,x}^1$ for $\tau \geq \frac{1}{2}$. Hence it will always surround $g(x)$.

Transitioning between solutions

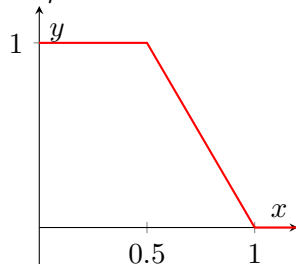Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following piecewise affine function:

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Is δ continuous?

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Is δ continuous? Note that if $\tau \to 1$ then $\delta_{\tau,x}^1$ will move along loop $\gamma_{0,x}^{\rho(\tau)t}$ at a speed that tends to $+\infty$, so we need to show that $\gamma_{0,x'}^{\rho(\tau)t'}$ tends uniformly to the constant loop as $(x', \tau, t') \to (x, 1, t)$.
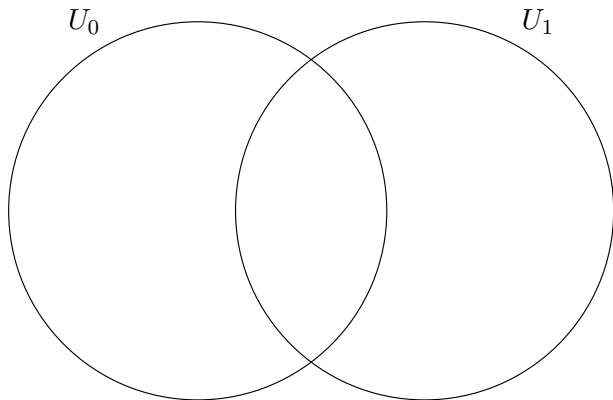
Transitioning between solutions

```
lemma satisfied_or_refund (γ₀ γ₁ : E → ℝ → loop F)
  (h₀ : γ₀ ∈ 𝓛 U) (h₁ : γ₁ ∈ 𝓛 U) :
  ∃ δ : ℝ → E → ℝ → loop F,
    (∀ τ, δ τ ∈ 𝓛 U) ∧
    δ 0 = γ₀ ∧
    δ 1 = γ₁ ∧
    continuous ↿δ
```

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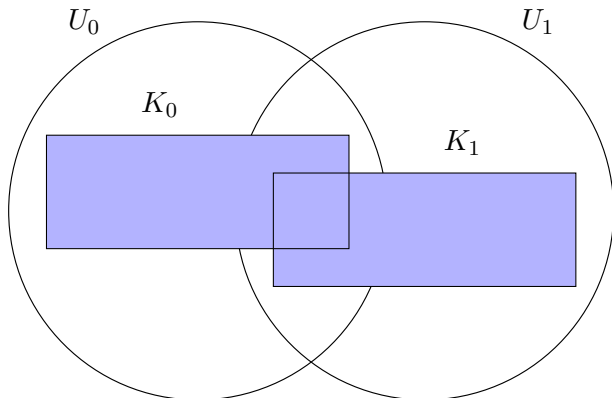Pick $K_i \subset U_i$ compact.

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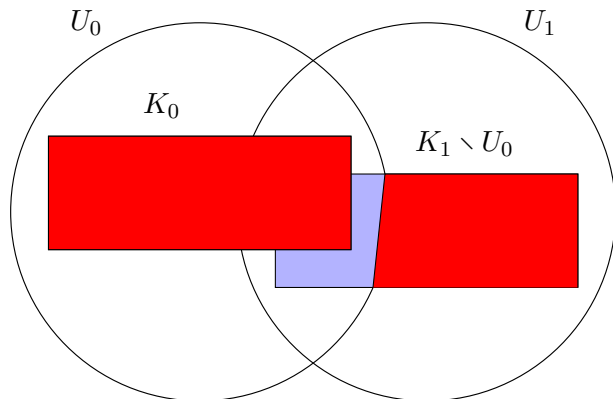Pick $K_i \subset U_i$ compact.
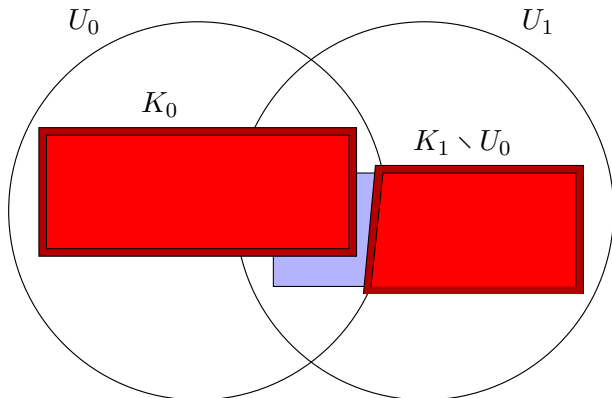
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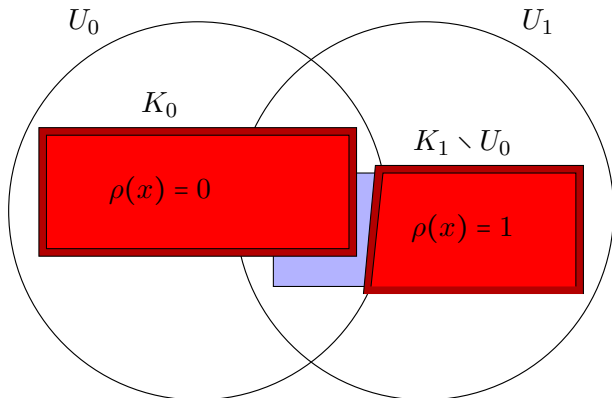Pick $K_i \subset U_i$ compact.

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Pick $K_i \subset U_i$ compact.

Let δ be a homotopy between γ_0 and γ_1 on $U_0 \cap U_1$. Then $\gamma_x = \delta_{\rho(x),x}$ is a solution in the shaded region.

Gluing solutions

We have shown the following result.

Proposition

If we have solutions $\gamma_i \in \mathcal{L}(U_i)$ with U_i an open neighborhood of the compact set K_i for $i \in \{0, 1\}$ then we can find a solution $\gamma \in \mathcal{L}(U)$ for some $U \in \mathcal{N}(K_0 \cup K_1)$ which coincides with γ_0 near K_0.

Induction

Lemma

If $\gamma^0 \in \mathcal{L}(U_0)$ with U_0 a neighborhood of K, then we can find $\gamma \in \mathcal{L}(E)$ which agrees with γ^0 near K.

Proof sketch.

We can find U_i, $i \geq 1$ be a locally finite family of open sets with local surrounding families of loops γ^i and compact subsets $K_i \subseteq U_i$ covering E. By repeatedly gluing these loops to γ^0, we obtain a sequence of families that is eventually constant in each K_i, so we obtain γ in the limit. $\qquad \square$

Smoothing

Now that we have a global continuous solution, we need to find a smooth approximation that preserves the following properties:

- γ_x^t is a loop based at $\beta(x)$
- $\gamma_x^0(s) = \beta(x)$
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We could take the convolution with a smooth bump function.

But then we need an argument using partition of unity to ensure that the family is equal to β on part of its domain.

Smoothing

In fact, we can do it without convolution, and using partition of unity in the following form:

Lemma

If $f : E \to F$ and $\varepsilon : E \to \mathbb{R}_{>0}$ are continuous then there is a C^∞ function $g : E \to F$ with $d(f(x), g(x)) < \varepsilon(x)$ for all x. Moreover, if f is smooth near a closed set C, then we can choose g such that $g(x) = f(x)$ for $x \in C$.

Using this lemma we can get all desired properties for the smooth approximation.

Reparametrization

Finally, we need to reparametrize the loop so that its average is exactly $g(x)$.

Convolution

I formalized the following version of the convolution of two functions.

Given $f : E \to \mathbb{R}$ and $g : E \to F$ the convolution $f \star g : E \to F$ is given by

$$(f \star g)(x) = \int_E f(t)g(x - t)d\mu(t).$$

Here E is an topological abelian group equipped with a Haar measure μ and F is a Banach space.

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This version is useful to smooth a vector-valued function with a bump function (bump functions were defined by Yuri Kudryashov in Lean).

- If f_n is a sequence of bump functions that tends to δ_{x_0} then $(f_n \star g)(x) \to g(x_0)$;
- If f is C^n and has compact support and g is locally integrable then $f \star g$ is C^n.

Convolution

Interestingly, all books that I could find that proves the second property, uses it by computing the partial derivatives of $f \star g$:

$$\frac{\partial}{\partial x_i}(f \star g) = \frac{\partial f}{\partial x_i} \star g.$$

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How to state this? $D_x(f \star g)$ is a linear map $E \to F$ and it should be a convolution of Df with g.

However, the convolution of $Df : E \to E^*$ and $g : E \to F$ is not in the form we defined before: neither of them is a scalar.

Convolution

Given $f : E \to F_1$ and $g : E \to F_2$ and a continuous bilinear map
$L : \operatorname{Hom}(F_1 \times F_2, F)$ we define the convolution $f \star_L g : E \to F$ is given by

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To compute the derivative, let

$$L' : \mathrm{Hom}(\mathrm{Hom}(E, F_1) \times F_2, \mathrm{Hom}(E, F))$$

given by $L'(M, y) = L(M(-), y)$.

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This has the regular properties of convolution.
To compute the derivative, let

$$L' : \mathrm{Hom}(\mathrm{Hom}(E, F_1) \times F_2, \mathrm{Hom}(E, F))$$

given by $L'(M, y) = L(M(-), y)$.

Theorem

If f is C^1 with compact support and g is locally integrable, then

$$D(f \star_L g) = Df \star_{L'} g.$$

Convolution

Proof.

We compute

$$
\begin{aligned}
D_{x_0}(f \star_L g) &= D_{x_0} \int_E L(f(x - t), g(t)) d\mu(t) \\
&= \int_E D_{x_0} L(f(x - t), g(t)) d\mu(t) \\
&= \int_E L(-, g(t)) \circ D_{x_0 - t} f \, d\mu(t) \\
&= \int_E L'(D_{x_0 - t} f, g(t)) d\mu(t) \\
&= (Df \star_{L'} g)(x_0).
\end{aligned}
$$

$\square$

Convolution

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\end{aligned}
$$

Since f has compact support, the integral is dominated by an integrable function, so we are allowed to pull the derivative inside the integral. □

Some Lean code

```
def convolution (f : E → F₁) (g : E → F₂)
  (L : F₁ →L[𝕂] F₂ →L[𝕂] F) (μ : measure E) :
  E → F :=
λ x, ∫ t, L (f t) (g (x - t)) ∂μ
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```
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lemma cont_diff_convolution_left
  (hcf : has_compact_support f) (hf : cont_diff 𝕂 n f)
  (hg : locally_integrable g μ) :
  cont_diff 𝕂 n (f ⋆[L, μ] g)
```

Conclusion

Proof assistants can be used to formalize tricky arguments in differential topology.

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To formalize we need to give a lot more details than drawing a picture.

Lean can help motivate interesting generalizations.

More Lean

```
lemma convolution_tendsto_right {l : filter ι}
  {φ : ι → E → ℝ} (hnφ : ∀ i x, 0 ≤ φ i x)
  (hiφ : ∀ i, ∫ s, φ i s ∂μ = 1)
  (hφ : tendsto (λ n, support (φ n)) l (𝒩 0).small_sets)
  (hmg : ae_strongly_measurable g μ)
  {x₀ : E} (hcg : continuous_at g x₀) :
  tendsto (λ i, (φ i ⋆[lsmul ℝ ℝ, μ] g) x₀) l (𝒩 (g x₀))
```