Math 1B

Power series

Mar 11 2020

1. For what values of x is the series $\sum_{n=1}^{\infty} n^n x^n$ convergent or divergent?

Root test: $\lim_{n\to\infty} |n^n x^n|^{1/n} = \lim_{n\to\infty} n|x|$

if $|x|=0$, then $= \lim_{n\to\infty} n\cdot 0 = 0 \to$ converges

if $|x|\neq 0$, then $= \lim_{n\to\infty} n|x| = \infty \to$ diverges

interval of convergence $= \{0\} \left(= [0,0]\right)$

2. Find the radius of convergence and interval of convergence:

(a) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ $\quad a_n = \frac{(x-1)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}}\right| = \lim_{n\to\infty}\frac{|x-1|n}{n+1} = |x-1|$$

Converges if $|x-1|<1$. $\rightsquigarrow -1 < x-1 < 1 \rightsquigarrow 0<x<2$

@ $x=0$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges $\quad$ @ $x=2$, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. $\boxed{[0,2)}$ $R=1$

(b) $\sum_{n=2}^{\infty}\frac{(x-2)^n}{n-1}$

$$\lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)-1}}{\frac{(x-2)^n}{n-1}}\right| = \lim_{n\to\infty}\frac{|x-2|(n-1)}{n} = |x-2|$$

...

$m=n-1 \rightsquigarrow \sum_{m=1}^{\infty}\frac{(x-2)^{m+1}}{m} = (x-2)\sum_{m=1}^{\infty}\frac{(x-2)^m}{m}$ $\quad x=y+1$

$= (y-1)\sum_{m=1}^{\infty}\frac{(y-1)^m}{m}$ $\quad 0\le y<2$

$0\le x-1<2$

$1\le x<3$ $\quad \boxed{[1,3)}$

(c) $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$ $\quad \frac{1}{n!} = \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\ \cdots\ \cdot\frac{1}{(n-1)}\cdot\frac{1}{n}$

$$\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{(n+1)!}}{\frac{(x-1)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)\,n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x-1}{n+1}\right| = \lim_{n\to\infty}\frac{|x-1|}{n+1} = 0$$

Since $0<1$, by ratio test, the series converges (indep. of what x is) so $\boxed{(-\infty,\infty)}$ $\quad R=\infty$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{-(-1)^n x^n}{n 5^n} = -\sum_{n=1}^{\infty} \frac{(-x/5)^n}{n}$

Recall: $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges if $z \in [-1, 1)$

with $z = -x/5$, series converges if $-1 \leq -\frac{x}{5} < 1$

$-5 \leq -x < 5$

$5 \geq x > -5$

$\boxed{(-5, 5],\ R=5}$

(e) $\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = \frac{x}{1} + \frac{x^2}{1 \cdot 3} + \frac{x^3}{1 \cdot 3 \cdot 5} + \frac{x^4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

$$\lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)}}{\frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}} \right| = \lim_{n \to \infty} \left| \frac{x}{2n+1} \right| = 0$$

$0 < 1$, so by ratio test, converges.

$\boxed{R = \infty \quad (-\infty, \infty)}$

__Theorem__ For $\sum_{n=0}^{\infty} c_n (x-a)^n$, these are the possibilities for the interval of convergence:

1) $\{a\}$ (radius 0)
2) $(-\infty, \infty)$ (radius ∞)
3) $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$ (radius R)

__Theorem__ A power series absolutely converges in the interior of its interval of convergence.

__Theorem__ If $\lim_{n \to \infty} |c_n|^{1/n} = c \neq 0$, then the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is $1/c$.

__Theorem__ If $\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = R$ exists, then the radius of convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$ is R.