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LETTER

Combining Parallel Linkage Construction and Echelon-Ferrers Construction for Constant-Dimension Codes

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SUMMARY In the constant dimension codes (CDCs) case, to explore the largest possible value $\mathcal{A}_q(n,d,k)$ is one of the most fundamental topics in subspace coding. Echelon-Ferrers is an approach proposed in 2009 for CDCs' construction, and has received extensive attentions. In this letter, in order to further improve the constructions, we study the method which combines linkage construction and echelon-Ferrers construction. This allows us to produce some improvements to lower bounds for constant dimension codes, including $\mathcal{A}_q(13,4,4)$,$\mathcal{A}_q(17,4,4)$, $\mathcal{A}_q(19,6,6)$ etc.
key words: *network coding; subspace code; constant dimension code; echelon-Ferrers Construction*

1. Introduction

R. Köetter and F. R. Kschischang proposed random linear network coding to encode information into a subspace of an ambient space [1]. This method improves the network flow with unknown topology by allowing any linear combination of packets at nodes, leading to the development of subspace coding.

Formally, let $\mathbb{F}_q$ be a finite field with q elements. The set of all subspaces of $\mathbb{F}_q^n$ is represented by $\mathcal{P}_q(n)$, and $\mathcal{P}_q(n)=\bigcup_{k=0}^{n}\mathcal{G}_q(k,n)$, where $\mathcal{G}_q(k,n)$ denotes the set of all k-dimensional subspaces of $\mathbb{F}_q^n$. With the following distance $d_S(U_1,U_2)$, all subspaces form a metric space:

$$d_S(U_1,U_2)=\dim(U_1+U_2)-\dim(U_1\cap U_2), \forall U_1,U_2\in\mathcal{P}_q(n).$$

It is one of the most fundamental problems to determine the maximum possible size $\mathcal{A}_q(n,d,k)$ of CDCs in subspace coding. A set $\mathbb{C}\subseteq\mathcal{G}_q(k,n)$ in $\mathbb{F}_q^n$ is called a $(n,|\mathbb{C}|,d,k)_q$ constant dimension code (abbreviated with CDC), if its distance of $\mathbb{C}$ is lower bounded by $d=\min\{d_S(U_1,U_2)\mid U_1,U_2\in\mathbb{C},U_1\neq U_2\}$. Often than not, the maximum possible value of an $(n,|\mathbb{C}|,d,k)_q$ CDC is usually denoted as $\mathcal{A}_q(n,d,k)$.

In this letter, we consider to combine parallel linkage construction and echelon-Ferrers to improve lower bounds for $\mathcal{A}_q(n,d,k)$. Towards this, we introduce a new concept called restricted echelon-Ferrers diagram, deriving from the echelon-Ferrers diagram, which leads to some improved constructions for CDCs. Our construction generalizes and contains the sub-code construction given in Corollary 4.5 [2] as a specific case.

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2. Previous Results

The construction for constant dimension codes can be achieved using a variety of approaches, and numerous methods for CDCs were invented in the literature, see [2]–[7], etc. The upper and lower bounds for $\mathcal{A}_q(n,d,k)$ are summarized at the web site `http://subspacecodes.uni-bayreuth.de` and the corresponding survey [8]. For the parameters cover $4\leq n\leq 19$, $q\in\{2,3,4,5,7,8,9\}$, $2\leq\frac{d}{2}\leq k$, $2\leq k\leq\frac{n}{2}$.

One of the most prominent construction methods is to use maximum rank distance (MRD) codes, which add an identity matrix I_k as the prefix: $\Im(I_k,Q(n,k,\frac{d}{2}))$. The codes obtained by this method we refer to lifted MRD code [1]. The *echelon-Ferrers* construction was later developed, which generalized the MRD codes by introducing a new class code of *Ferrers diagram rank-metric codes* [6]. Another powerful construction are the linkage construction [9] and its generalization [10]. Constructing new CDCs from two versions of lifted MRD codes were first proposed in [7]. This simple and effective construction produces a large amount of follow-up improvements, see the papers [2], [4], [11], [12] and [13], which finally led to the improvement of linkage construction. [14] and [13] each independently proposed a construction from two versions of linkage construction. [2] studied several approaches to combine subspace code. Shuangqing Liu, Lijun Ji etc. [15], [16], and [17] generalized the multilevel construction and inverse multilevel construction by introducing bilateral/double identifying vectors. The most surprising work in recent years is the mixed dimension construction [18] and its improvements [19].

3. Echelon-Ferrers Construction and its Variant

In this section, we will introduce a new concept called restricted echelon-Ferrers diagram, which leads to improved construction for CDCs.

3.1 Ferrers Diagram

Throughout the letter, we will use the following notation, and refer to [6] for the details.

Any k-dimensional subspace $X \in \mathcal{G}_q(k,n)$, can be uniquely expressed as its reduced row echelon (RRE) form, denoted $E(X)$. The *identifying vector* of X, denoted $v(X)$, is a binary vector of length n and weight k, where the k *ones* in $v(X)$ are located at the pivots of $E(X)$. Each pivot in $E(X)$ has a leading coefficient. To construct the Ferrers diagram of X, first delete all zeros in each row of $E(X)$, which lies on the left side of the pivot, then remove the columns having the pivots, and shift all the remaining entries to the right. The resulting structure is called the Ferrers tableaux form of X, and is denoted by $\mathcal{F}(X)$. Finally, replacing the entries in $\mathcal{F}(X)$ with dots yields the Ferrers diagram of X.

Example 1. *Let $v(X)$ be 11001100, we have the following generator matrix in its RRE form:*

$$E(X) = \begin{pmatrix} 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

Evidently, its Ferrers diagram $\mathcal{F}$ is given by

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & \bullet & \bullet \end{array}.$$

For any two matrices $A, B \in \mathbb{F}_q^{m\times\ell}$, $d_R(A,B) := \mathrm{rank}(A-B)$ is the rank-metric between A and B on $\mathbb{F}_q^{m\times\ell}$, where $\mathbb{F}_q^{m\times\ell}$ is the set of all $m \times \ell$ matrices over the field $\mathbb{F}_q$. A subset $\mathbb{C} \subset \mathbb{F}_q^{m\times\ell}$ with the rank-metric d_R is called a rank-metric code. The rank-distance of a rank-metric code $\mathbb{C}$ is defined as $d_R(\mathbb{C}) := \min\{d_R(A,B) : A, B \in \mathbb{C}, A \neq B\}$. It is well-known that the number of codewords in $\mathbb{C}$ is upper bounded by $q^{\max\{m,\ell\}\times(\min\{m,\ell\}-d+1)}$. A code reaching this upper bound is called a maximum rank-distance (MRD) code (denoted by $Q_q(m,\ell,d)$).

Suppose $\mathcal{F}$ is a Ferrers diagram with the shape $\ell \times m$: ℓ dots in the top row and m dots in the rightmost column. A linear rank-metric code $\mathbb{C}_{\mathcal{F}} \subset \mathbb{F}_q^{m\times\ell}$ is called a Ferrers diagram rank-metric (FDRM) code if $\forall A \in \mathbb{C}_{\mathcal{F}}$, elements of A which are not in $\mathcal{F}$ are zeros. A lifted FDRM codes can be obtained by substituting each codeword $A \in \mathbb{C}_{\mathcal{F}}$ in the columns of $E(X)$ which correspond to the zeros of X. We can get an upper bound from the size of $\dim(\mathbb{C}_{\mathcal{F}})$ according to Theorem 1.

Theorem 1. *([6]) Suppose $\mathcal{F}$ is the Ferrers diagram with the shape $\ell \times m$ and $\mathbb{C}_{\mathcal{F}} \subseteq \mathbb{F}_q^{m\times\ell}$ is the corresponding FDRM code that satisfies $\mathrm{rank}(A-B) \geq d$ for any $A, B \in \mathbb{C}_{\mathcal{F}}$ with $A \neq B$. Then the dimension of code $\mathbb{C}_{\mathcal{F}}$ has an upper bound $\dim(\mathbb{C}_{\mathcal{F}}) \leq \min_i\{w_i\}$. Here, w_i is the total number of dots in $\mathcal{F}$, which are neither included in the rightmost $\delta-1-i$ columns, nor included in the first i rows for $0 \leq i \leq \delta-1$.*

Paper [6] proved that when $d = 2$ or 4, the upper bound of Theorem 1 can be reached, and in the meanwhile, they conjectured that for all appropriate parameters, the upper bounds can be attained.

3.2 Restricted Ferrers Diagram

Let $\mathcal{F}$ be a Ferrers diagram, if in addition the entities in the upper right corner of $\mathcal{F}$ with the size $(a \times b)$ are all zeros, we augment the notation to $\mathcal{F}_{a\times b}$ and refer to it as restricted Ferrers diagram. Such a linear rank-metric code $\mathbb{C}_{\mathcal{RF}}$ of $\mathbb{F}_q^{m\times\ell}$ is called a restricted Ferrers diagram rank-metric (RFDRM) code, if all entries of M not in $\mathcal{F}_{a\times b}$ are zeros. A lifted RFDRM codes is obtained by substituting each codeword $A \in \mathbb{C}_{\mathcal{RF}}$ in the columns of $E(X)$ which correspond to the zeros of X.

Example 2. *Continue the above example, let $(a\times b)$ be (2×2), then we have the generator matrix in its RRE form is*

$$E(X) = \begin{pmatrix} 1 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 1 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

Clearly, its restricted Ferrers diagram is given as follow:

$$\begin{array}{cccc} \bullet & \bullet & & \\ \bullet & \bullet & & \\ & & \bullet & \bullet \\ & & \bullet & \bullet \end{array},$$

Now the problem is to determine the upper bound on the size of $\dim(\mathbb{C}_{\mathcal{RF}})$. When $\mathbb{C}_{\mathcal{RF}}$ is divided into two parts, the upper bound is at least the product of the two parts.

3.3 Delsarte Theorem

The rank distribution of a linear MRD code $Q_q(m,n,d)$ $(m \geq n)$ is defined as $\mathcal{A}_r(Q_q(m,n,d)) = |\{M \in Q, rank(M) = i\}|$, where $n \leq r \leq m$ (see [20], [21]). The rank distribution of MRD codes is entirely determined by its own parameters: (q,r,m,n,d).

Theorem 2. *([21] and [20]) W.L.O.G, let $m \geq n$, given a linear MRD code $Q_q(m,n,d)$ with distance d, then its rank distribution is determined by $\mathcal{A}_r(Q) = \binom{n}{r}_q \sum_{i=0}^{r-d}(-1)^i q^{\binom{i}{2}} \binom{r}{i}_q \left(\frac{q^{m(n-d+1)}}{q^{m(n+i-r)}} - 1\right)$, where $\mathcal{A}_r(Q)$ represents the size of elements in $Q_q(m,n,d)$ with rank r, and $n \leq r \leq m$. Here, $\binom{r}{i}_q = \prod_{j=0}^{k-1} \frac{q^{r-j}-1}{q^{i-j}-1}$.*

3.4 Parallel Linkage Construction

Let us review some common symbols in [9]. For any set $\mathcal{U} \subset \mathbb{F}_q^{k\times n}$ is an SC-representation set, if 1) $\forall U \in \mathcal{U}$, $\mathrm{rank}(U) = k$, and 2) $\forall U_1, U_2 \in \mathcal{U}$, if $U_1 \neq U_2$, then $\Im(U_1) \neq \Im(U_2)$. Given any matrix $M \in \mathbb{F}_q^{k\times \ell}$, we denote $\Im(M)$ as the row space spanned by M.

Proposition 1. *([9]) Suppose that U is an SC-representation set, and is an $(n_1, N_1, d_1, k)_q$ constant dimension code, and $\mathcal{M} \subset \mathbb{F}_q^{k\times n_2}$ is a linear rank-metric code with distance d_2 and contains N_2 elements. Then $\mathbb{C}_1 = \{\Im(U|M) : U \in \mathcal{U}, M \in \mathcal{M}\}$ is an $(n_1 + n_2, N_1N_2, \min\{d_1, 2d_2\}, k)_q$ constant dimension code. Here, $(U|M)$ is a $k \times (n_1 + n_2)$ matrix formed by concatenating U and M.*

Let $\mathbb{C}_2 = \{\Im(Q_{21}|U_{22}) : Q_{21} \in \mathbf{Q}, U_{22} \in \mathbf{U}\}$, where $\mathbf{Q} \subset Q_q(n_1, k, d_2)$ is a linear rank-metric code with rank distance d_2 and the size N_3, and $\mathbf{U}$ is an SC-representation set of an $(n_2, N_4, d_1, k)_q$ constant dimension code. Similarly, $\mathbb{C}_2$ is another $(n_2 + n_1, N_3N_4, \min\{d_1, 2d_2\}, k)_q$ constant dimension code because of the symmetry.

To calculate the cardinality of $\mathbb{C}_1 \cup \mathbb{C}_2$, we refer to the following theorem (Theorem 3 in [22]).

Theorem 3. *([22]) Let $k \geq d$, $n_1 \geq k$, $n_2 \geq k$, $0 \leq t \leq n_1 - k$, we have $\mathcal{A}_q(n_1 + n_2, k, d) \geq \left|Q_q(n_1, k, \frac{d}{2})\right| \times \mathcal{A}_q(n_2, k, d) + \mathcal{A}_q(n_1 - t, k, d) \times \left(1 + \sum_{r=\frac{d}{2}}^{k-\frac{d}{2}} \mathcal{A}_r\left(Q_q(n_2 + t, k, \frac{d}{2})\right)\right)$.*

4. Our Construction

In this section, our main results, i.e the improved construction for CDCs are established, which generalizes the sub-code construction given in Corollary 4.5 [2].

4.1 General Construction

More precisely, our construction is based on the above parallel linkage construction by modifying the echelon-Ferrers construction. In order to preserve the distance, we propose a new notion of restricted Ferrers diagram code, that the codes constructed in Corollary 4.5 [2] is a part of it.

To briefly describe the construction, we demonstrate our construction as the following theorem.

Theorem 4. *Let $\mathcal{U}_i \subseteq \mathbb{F}_q^{k\times n_i}$ be SC-representing sets with cardinality N_i, $d_S(\mathcal{U}_i) = d$, and $n_i \geq k$, $k \geq d$. Assume that $\mathbb{C}_{R_i} \subseteq \mathbb{F}_q^{k\times n_i}$ are linear rank-metric codes with $|\mathbb{C}_{R_i}| = N_{R_i}$ and $d_R(\mathbb{C}_{R_i}) = \frac{d}{2}$. Here, $1 \leq i \leq 2$. In addition, the rank of each element in $\mathbb{C}_{R_2}$ is at most $k - \frac{d}{2}$.*

Let the identifying vectors v_j with length $n := n_1 + n_2$ and weight k satisfy the following properties for $j = 1, 2, \cdots$.

(a) For each vector v_j, the count of 1's in the first n_1 positions and the last n_2 positions are both greater than or equal to $\frac{d}{2}$.

(b) For any two distinct vectors v_{j_1} and v_{j_2}, the Hamming distance $H(v_{j_1}, v_{j_2}) \geq d$.

Let $\mathbb{C}_{\mathcal{RF}_j} \subseteq \mathbb{F}_q^{k\times(n-k)}$ be a restricted FDRM code and $d_R(\mathbb{C}_{\mathcal{RF}_j}) = \frac{d}{2}$, where $\mathcal{RF}_j$ is a restricted Ferrers diagram corresponding to the identifying vector v_j with the size $(a \times b)$=$(n_2 \times \frac{d}{2})$.

Denote by $\mathbb{C}$ the constant dimension code of length $n = n_1 + n_2$ as $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2 \cup \mathbb{C}_3$, where

$\mathbb{C}_1 = \{\Im(U|M) \mid U \in \mathcal{U}_1, M \in \mathbb{C}_{R_1}\}$;

$\mathbb{C}_2 = \{\Im(M|U) \mid U \in \mathcal{U}_2, M \in \mathbb{C}_{R_2}\}$;

$\mathbb{C}_3 = \cup_j \mathbb{C}_{\mathcal{RF}_j}$, $\mathbb{C}_{\mathcal{RF}_j}$ is the lifted RFDRM code of $\mathbb{C}_{\mathcal{RF}_j}$.

Thus, $\mathbb{C}$ is an $(n, N, d, k)_q$ CDC with $N = N_1 \cdot N_{R_1} + N_2 \cdot N_{R_2} + \sum_j |\mathbb{C}_{\mathcal{RF}_j}|$.

Proof. The proof consists of two parts:

1) Note that the pivots of $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3$ are in the first n_1 positions, the last n_2 position, parts in the first n_1 positions and parts in the last n_2 positions, respectively. Therefore, $\mathbb{C}_1 \cap \mathbb{C}_3 = \emptyset, \mathbb{C}_2 \cap \mathbb{C}_3 = \emptyset$, and $\mathbb{C}_1 \cap \mathbb{C}_2 = \emptyset$. Hence, the size of the code $\mathbb{C}$ is $N = N_1 \cdot N_{R_1} + N_2 \cdot N_{R_2} + \sum_j |\mathbb{C}_{\mathcal{RF}_j}|$.

2) It has been proved in [22] Theorem 3 that the distance $d_S(W_1, W_2) \geq d$ for any subspace $W_1 \in \mathbb{C}_1, W_2 \in \mathbb{C}_2$. According to the definition, $\mathbb{C}_{\mathcal{RF}_j}$ is a CDC with $d_S(\mathbb{C}_{\mathcal{RF}_j}) \geq d$. Herein, we need to prove that for any subspace $W_1 \in \mathbb{C}_1, W_2 \in \mathbb{C}_2, W_3 \in \mathbb{C}_3$, we always have $d_S(W_1, W_3) \geq d, d_S(W_2, W_3) \geq d$.

For any identifying vector v_j $(j = 1, 2, \cdots)$, we note that this vector has $\frac{d}{2}$ ones in the last n_2 positions, and can be illustrated in reduced row echelon form as follows:

$$Z_1 := \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}_{k\times n},$$

where Z_{11} with the size of $(k - \frac{d}{2}) \times n_1$, Z_{12} and Z_{21} are zero matrices with the size of $(k - \frac{d}{2}) \times n_2$ and $\frac{d}{2} \times n_1$, respectively, Z_{22} with the size of $\frac{d}{2} \times n_2$.

It is clear that $W_1 = \Im(U_1|M_1)$, $U_1 \in \mathcal{U}_1, M_1 \in \mathcal{M}_1$, $W_2 = \Im(M_2|U_2)$, $U_2 \in \mathcal{U}_2, M_2 \in \mathcal{M}_2$ then

$$Z_2 := \begin{pmatrix} U_1 & M_1 \\ M_2 & U_2 \\ Z_{11} & 0_{(k-\frac{d}{2})\times n_2} \\ 0_{\frac{d}{2}\times n_1} & Z_{22} \end{pmatrix}_{3k\times n}.$$

Notice that the elements in matrix Z_{21} are all 0, and the rank of U_1 is k, hence, the rank of the matrix Z_2 is at least $k + \frac{d}{2}$, therefore, $d_S(W_1, W_3) \geq d$. Consider that Z_{12} are zero matrix, by the same way, the distance $d_S(W_2, W_3) \geq d$ can be also proved.

□

Example 3. *We apply the above theorem to construction $\mathcal{A}_2(10, 4, 5)$, it is clear that $n = 10, d = 4, n_1 =$*

$n_2 = 5, k = 5$. *Therefore,* $|\mathbb{C}_1| = 1048576$ *with the identifying vector (111110000),* $|\mathbb{C}_2| = 129736$ *with the identifying vector (0000011111). To compute the size of the lifted RFDRM code of all identifying vectors as illustrated in Table 1, we have* $\mathcal{A}_2(10,4,5) \geq 1048576 + 129736 + 64 + 2 \times 2^4 + 6 \times 2^3 + 2 \times 2^2 + 5 = 1178469$.

Table 1 Construction for $\mathcal{A}_q(10,4,5)$

	Identifying Vector	Dim		Identifying Vector	Dim
1	1110011000	6	9	1100101100	4
2	1101010100	4	10	1010110100	3
3	1110000110	3	11	0111010001	0
4	1001111000	3	12	1010101010	2
5	1101001010	3	13	0110101001	0
6	1011010010	2	14	0101100101	0
7	1100110010	3	15	0101100101	0
8	1011001100	3	16	0011100011	0

From the above example, we can adopt Corollary 4.5 in [2] to improve the size of the RFDRM code. We found that the RFDRM code with corresponding identifying vector (1110011000) happens to be the following Corollary 1. Then $\mathcal{A}_2(10,4,5) \geq 1178917$, which exceeds the current best bounds 1178824. This holds for the cases $\mathcal{A}_q(n,4,5)$ when n varies from 15 to 19, $q \in \{2,3,4,5,7,8,9\}$.

Corollary 1 (Corollary 4.5 [2]). *Let* $n = \sum_{i=1}^{l} n_i$, $\sigma_i = \sum_{j=1}^{i} n_j, 1 \leq i \leq l, \sigma_0 = 0$. *Let* E_i *denote the* $(n - n_i)$*-subspace of* $\mathbb{F}_q^n$ *consisting of all vector in* $\mathbb{F}_q^n$ *that have zeros for the coordinates between* $\sigma_{i-1}+1$ *and* σ_i *for all* $1 \leq i \leq l$. $\sum_{i=1}^{l} a_i = k, \sum_{i=1}^{l} b_i = k - \frac{d}{2}$. *Let* D_i^j *be an* $(n_i, d, n_i - a_i)$*-CDC, and* $\bar{D}_i^j$ *be an embedding of* D_i^j *in* $\mathbb{F}_q^n$ *such that the vectors contained in an element of* D_i^j *are in* E_i, $1 \leq i \leq l$. *Define the code* $\mathbb{D}$ *as follows:* $\mathbb{D} = \cup_{j=1}^{s}\{U_1 \times U_2 \times \cdots \times U_l : U_h \in \bar{D}_h^j, 1 \leq \forall h \leq l\}$.

On the basis of Theorem 3, we can achieve D *with the size of* $|\mathbb{D}| \geq s \cdot \prod_{i=1}^{l}\left(Q_q(a_i, n_i - a_i, \frac{d}{2})\right)$, *where* $\alpha_i = (Q_q(a_i, n_i - a_i, a_i - b_i)) / \left(Q_q(a_i, n_i - a_i, \frac{d}{2})\right)$, *and* $s = \min\{\alpha_i : 1 \leq i \leq l\}$.

Remark: It is interesting to discover that our construction generalizes the sub-code construction proposed in Corollary 4.5 [2]. Our approach improves upon this by modifying the echelon-Ferrers structure to meet the distance constraint, ultimately resulting in a larger number of codewords.

4.2 Examples

Let $n = 12, n_1 = 6, n_2 = 6, k = 6, d = 6$, apply the above theorem to construction $\mathcal{A}_2(12,6,6)$. Therefore, $|\mathbb{C}_1| = q^{24}$ with the vector (111111000000), $|\mathbb{C}_2| = 1 + \sum_{r=3}^{3} \mathcal{A}_r(Q_q(6,6,3))$. We calculate the size of the lifted RFDRM code of the remaining 20 identifying vectors, and $\mathcal{A}_q(12,6,6) \geq q^{24} + 1 + \sum_{r=3}^{3} \mathcal{A}_r(Q_q(6,6,3)) + q^9 + 4 \times q^2 + 3 \times q + 12$. Let $q = 2$, we have $\mathcal{A}_2(12,6,6) \geq 16865649$, which exceeds the bounds in [2].

Similarly, we have $\mathcal{A}_q(15,4,5) \geq q^{40} + \mathcal{A}_q(10,4,5) \times (1 + \sum_{r=2}^{3} \mathcal{A}_r(Q_q(5,5,2))) + q^{20} + 3q^9 + 3q^8 + 5q^7 + 3q^6 + 8q^5 + 7q^4 + 4q^3 + 6q^2 + q + 3$. Let $q = 2$, we have $\mathcal{A}_2(15,4,5) \geq 1252448590381$, which exceeds the current best bounds 1252448586816.

We have $\mathcal{A}_q(16,4,5) \geq q^{44} + \mathcal{A}_q(11,4,5) \times (1 + \sum_{r=2}^{3} \mathcal{A}_r(Q_q(5,5,2))) + q^{22} + 3q^{10} + 3q^9 + 5q^8 + 3q^7 + 8q^6 + 6q^5 + 4q^4 + 5q^3 + 6q^2 + 4q + 2$. Let $q = 2$, then $\mathcal{A}_2(16,4,5) \geq 20021891632482$, which improves upon the corresponding results.

We have $\mathcal{A}_q(18,6,6) \geq \mathcal{A}_q(12,6,6) \times q^{24} + (1 + \sum_{r=3}^{3} \mathcal{A}_r(Q_q(12,6,3))) + q^{18} + q^9 + 5q^8 + 5q^7 + 9q^6 + 2q^5 + 5q^4 + 4q^3 + 7q^2 + 7q + 10$. $\mathcal{A}_q(19,6,6) \geq \mathcal{A}_q(12,6,6) \times q^{28} + (1 + \sum_{r=3}^{3} \mathcal{A}_r(Q_q(13,6,3))) + q^{21} + 2q^{10} + 3q^9 + 5q^8 + 6q^7 + 4q^6 + 8q^5 + 10q^4 + 11q^3 + 8q^2 + 7q + 12$, and $\mathcal{A}_q(16,4,4) \geq \mathcal{A}_q(12,4,4) \times q^{12} + (1 + \sum_{r=2}^{2} \mathcal{A}_r(Q_q(12,4,2))) + q^{14} + 3q^{10} + 4q^9 + 5q^8 + 3q^6 + 4q^5 + 5q^4 + 3q^2 + 2q + 6$.

These new bounds exceed the current theoretic bounds in [21], [7],[4],[23], [13], even the latest improvements in [2]. Limited by space, all these improvements are omitted here. Some result are coincide with the paper [24].

4.3 Improvements with $t = 1$

The above section deals with the case $t = 0$, some cases can be further improved by Theorem 3 with $t = 1$. More specifically, we embody Theorem 3 as follows:

Let $k \geq d, n_1 \geq k, n_2 \geq k, t = 1$, then $\mathcal{A}_q(n_1 + n_2, k, d) \geq \left|Q_q(n_1, k, \frac{d}{2})\right| \times \mathcal{A}_q(n_2, k, d) + \mathcal{A}_q(n_1 - 1, k, d) \times \left(1 + \sum_{r=\frac{d}{2}}^{k-\frac{d}{2}} \mathcal{A}_r\left(Q_q(n_2 + 1, k, \frac{d}{2})\right)\right)$.

With this, we have following bounds:

Let $n_1 = 8, n_2 = 5, d = 4, k - 4$, then $\mathcal{A}_q(13,4,4) \geq \mathcal{A}_q(8,4,4) \times q^{15} + (1 + \sum_{r=1}^{2} \mathcal{A}_r(Q_q(9,4,2))) + q^{12} + 3q^7 + 5q^6 + 3q^5 + q^4 + 6q^3 + 8q^2 + 7q + 4$. This bound exceeds the current theoretic bounds in [4], [13], [25], even the latest improvements in [2], [24].

By the same way, we have $\mathcal{A}_q(17,4,4) \geq \mathcal{A}_q(12,4,4) \times q^{15} + (1 + \sum_{r=1}^{2} \mathcal{A}_r(Q_q(13,4,2))) + q^{16} + 3q^{11} + 5q^{10} + 3q^9 + q^8 + 7q^7 + 9q^6 + 7q^5 + 5q^4 + 2q^3 + 5q^2 + 3q^1 + 1$, and $\mathcal{A}_q(19,6,6) \geq \mathcal{A}_q(12,6,6) \times q^{28} + (1 + \sum_{r=3}^{3} \mathcal{A}_r(Q_q(14,6,3))) + q^{21} + 2q^{10} + 3q^9 + 5q^8 + 6q^7 + 4q^6 + 8q^5 + 10q^4 + 11q^3 + 8q^2 + 7q + 12$.

5. Conclusion

In this letter, we presented new constructions for constant dimension codes. Several new lower bounds on $\mathcal{A}_q(n,d,k)$ can be obtained from theses constructions, which indicates these codes have better bounds than in the tables of [8]. Moreover, the expression of these bounds can be calculated by Theorem 4. From Corollary 1, we may wonder the RFDRM codes can be further improved, hence, it would be interesting to derive bounds better than the ones in Theorem 4.

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