A New Variation of Hat Guessing Games

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Abstract. Several variations of hat guessing games have been popularly discussed in recreational mathematics. In a typical hat guessing game, after initially coordinating a strategy, each of n players is assigned a hat from a given color set. Simultaneously, each player tries to guess the color of his/her own hat by looking at colors of hats worn by other players. In this paper, we consider a new variation of this game, in which we require at least k correct guesses and no wrong guess for the players to win the game, but they can choose to "pass". A strategy is called *perfect* if it can achieve the simple upper bound $\frac{n}{n+k}$ of the winning probability. We present sufficient and necessary condition on the parameters n and k for the existence of perfect strategy in the hat guessing games. In fact for any fixed parameter k, the existence of perfect strategy can be determined for every sufficiently large n.

In our construction we introduce a new notion: (d_1, d_2)-regular partition of the boolean hypercube, which is worth to study in its own right. For example, it is related to the k-dominating set of the hypercube. It also might be interesting in coding theory. The existence of (d_1, d_2)-regular partition is explored in the paper and the existence of perfect k-dominating set follows as a corollary.

Keywords: Hat guessing game; perfect strategy; hypercube; k-dominating set; perfect code

1 Introduction

Several different hat guessing games have been studied in recent years [1–8, 10, 11]. In this paper we investigate a variation where players can either give a guess or pass. It was first proposed by Todd Ebert in [4]. In a standard setting there are n players sitting around a table, who are allowed to coordinate a strategy before the game begins. Each player is assigned a hat whose color is chosen randomly and independently with probability $1/2$ from two possible colors, red and blue. Each player is allowed to see all the hats but his own. Simultaneously, each player guesses its own hat color or passes, according to their pre-coordinated strategy. If at least one player guesses correctly and no player guesses wrong, the players win the game. Their goal is to design a strategy to maximize their winning probability.

By a simple counting argument there is an upper bound on the maximum winning probability, $n/(n+1)$. It is known that this upper bound is achieved if and only if n has the form $2^t - 1$ [4]. It turns out that the existence of such perfect strategy that achieves the upper bound corresponds to the existence of a perfect 1-bit error-correcting code in $\{0, 1\}^n$.

In this paper, we present a natural generalization of Ebert's hat guessing problem: The setting is the same as in the original problem, every player can see all other hats except his own, and is allowed to guess or pass. However, the requirement for them to win the game is generalized to be that at least k players from them should guess correctly, and no player guesses wrong ($1 \le k \le n$). Note that when $k = 1$, it is exactly the original problem.

We denote by $P_{n,k}$ the maximum winning probability for the problem with n players when at least k correct guesses are required. Similarly to the $k = 1$ case, $P_{n,k}$ has a simple upper bound $P_{n,k} \le \frac{n}{n+k}$. We call a pair (n, k) *perfect* if this upper bound can be achieved, i.e. $P_{n,k} = \frac{n}{n+k}$. There is a simple necessary condition for a pair (n, k) to be perfect, and our main result states that this condition is almost sufficient:

Theorem 1. *For any $d, k, s \in \mathbb{N}$ with $s \geq k + \lceil \log k \rceil - 1$, $(d(2^s - k), dk)$ is perfect, in particular, $(2^s - k, k)$ is perfect.*

There exists pair (n, k) with the necessary condition but not perfect, see the remark in Section 4.

Here is the outline of the proof: first we give a general characterization of the winner probability $P_{n,k}$ by using the size of the minimum k-dominating set of the hypercube. Then we convert the condition of (n, k) perfect to some kind of *regular partition* of the hypercube (see the definition in Section 2). Our main contribution is that we present a strong sufficient condition for the existence of such partition, which nearly matches the necessary condition. Then we can transform it into a perfect hat guessing strategy.

As a corollary of Theorem 1, we also give asymptotic characterization of the value $P_{n,k}$. For example, we show that for any fixed k, the maximum winning probability approaches 1 as n tends to the infinity.

Related work:

Feige [5] considered some variations including the discarded hat version and the everywhere balanced version. Lenstra and Seroussi [8] studied the case that n is not of form $2^m - 1$, they also considered the case with multiple colors. In [3], Butler, Hajiaghayi, Kleinberg and Leighton considered the worst case of hat placement with sight graph G, in which they need to minimize the maximum wrong guesses over all hat placements. In [6] Feige studied the case that each player can see only some of other players' hats with respect to the sight graph G. In [10], Paterson and Stinson investigated the case that each player can see hats in front of him and they guess one by one. Very recently, Buhler, Butler, Graham and Tressler [2] studied the case that every player needs to guess and the players win the game if either exactly k_1 or k_2 players guess correctly, they showed that the simple necessary condition is also sufficient in this game.

The rest of the paper is organized as follows: Section 2 describes the definitions, notations and models used in the paper. Then, Section 3 presents the result of the existence of (d_1, d_2)-regular partition of hypercube while Section 4 shows the main result of the hat guessing game. Finally, we conclude the paper in Section 5 with some open problems. The perfectness of some small (n, k)'s are listed in the appendix.

2   Preliminaries

We use Q_n to denote the the n dimension boolean hypercube $\{0,1\}^n$. Two nodes are adjacent on Q_n if they differ by only one bit. We encode the blue and red color by 0 and 1. Thus the placement of hats on the n players' heads can be represented as a node of Q_n. For any $x \in Q_n$, $x^{(i)}$ is used to indicate the string obtained by flipping the ith bit of x. Throughout the paper, all the operations are over $\mathbb{F}_2$. We will clarify explicitly if ambiguity appears.

Here is the model of the hat guessing game we consider in this paper: The number of players is denoted by n and players are denoted by $p_1, \ldots, p_n$. The colors of players' hats will be denoted to be $h_1, \ldots, h_n$, which are randomly and independently assigned from $\{0, 1\}$ with equal probability. $h = (h_1, \ldots, h_n)$. Let $h_{-i} \in Q_{n-1}$ denote the tuple of colors $(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n)$ that player p_i sees on the others' heads. The strategy of player p_i is a function $s_i : Q_{n-1} \to \{0, 1, \bot\}$, which maps the tuple of colors h_{-i} to p_i's answer, where $\bot$ represents p_i answers "pass" (if some player answers pass, his answer is neither correct nor wrong). A strategy $\mathcal{S}$ is a collection of n functions $(s_1, \ldots, s_n)$. The players win the game if at least k of them guess correctly and no one guesses wrong. We use $P_{n,k}$ to denote the maximum winning probability of the players. The following two definitions are very useful in characterization $P_{n,k}$:

Definition 1. *A subset $D \subseteq V$ is called a k-dominating set of graph $G = (V, E)$ if for every vertex $v \in V \setminus D$, v has at least k neighbors in D.*

Definition 2. *A partition (V_1, V_2) of hypercube Q_n is called a (d_1, d_2)-regular partition if each node in V_1 has exactly d_1 neighbors in V_2, and each node in V_2 has exactly d_2 neighbors in V_1.*

For example, consider the following partition (V_1, V_2) of Q_3: $V_1 = \{000, 111\}$, and $V_2 = Q_3 \setminus V_1$. For each vertex in V_1, there are 3 neighbors in V_2, and for each vertex in V_2, there is exactly one neighbor in V_1. Thus (V_1, V_2) forms a $(3, 1)$-regular partition of Q_3.

3 (d_1, d_2)-Regular Partition of Q_n

In this section we study the existence of (d_1, d_2)-regular partition of Q_n.

Proposition 1. *Suppose $d_1, d_2 \leq n$, if there exists a (d_1, d_2)-regular partition of hypercube Q_n, then the parameters d_1, d_2, n should satisfy $d_1 + d_2 = \gcd(d_1, d_2)2^s$ for some $s \leq n$.*

Proof. Suppose the partition is (V_1, V_2), we count the total number of vertices

$$|V_1| + |V_2| = 2^n,$$

and the number of edges between two parts

$$d_1 |V_1| = d_2 |V_2| .$$

By solving the equations, we obtain

$$|V_1| = \frac{d_2}{d_1 + d_2}2^n, \quad |V_2| = \frac{d_1}{d_1 + d_2}2^n.$$

Both $|V_1|$ and $|V_2|$ should be integers, therefore $d_1 + d_2 = \gcd(d_1, d_2)2^s$ holds, since $\gcd(d_1, d_1 + d_2) = \gcd(d_2, d_1 + d_2) = \gcd(d_1, d_2)$.

Proposition 2. *If there exists a (d_1, d_2)-regular partition of hypercube Q_n, then there exists a (d_1, d_2)-regular partition of Q_m for every $m \geq n$.*

Proof. It suffices to show that the statement holds when $m = n + 1$, since the desired result follows by induction. Q_{n+1} can be treated as the union of two copies of Q_n (for example partition according to the last bit), i.e. $Q_{n+1} = Q_n^{(1)} \cup Q_n^{(2)}$. Suppose (V_1, V_2) is a (d_1, d_2)-regular partition of $Q_n^{(1)}$. We can duplicate the partition (V_1, V_2) to get another partition (V_1', V_2') of $Q_n^{(2)}$. Then we can see that $(V_1 \cup V_1', V_2 \cup V_2')$ forms a partition of Q_{n+1}, in which each node has an edge to its duplicate through the last dimension. Observe that each node in V_1 (V_1') still has d_1 neighbors in V_2 (V_2') and same for V_2 (V_2'), and the new edges introduced by the new dimension are among V_1 and V_1', or V_2 and V_2', which does not contribute to the edges between two parts of the partition. Therefore we constructed a (d_1, d_2)-regular partition of Q_{n+1}.

Proposition 3. *If there exists a (d_1, d_2)-regular partition of Q_n, then there exists (td_1, td_2)-regular partition of Q_{tn}, for any positive integer t.*

Proof. Suppose (V_1, V_2) is a (d_1, d_2)-regular partition of Q_n. Let $x = x_1 x_2 \cdots x_{nt}$ be a node in Q_{nt}. We can divide x into n sections of length t, and denote the sum of ith section by w_i, i.e.

$$w_i(x) = \sum_{j=ti-t+1}^{ti} x_j, \quad (1 \leq i \leq n).$$

Let $R(x) = w_1(x)w_2(x) \ldots w_n(x) \in Q_n$. Define

$$V_i' = \{x \in Q_{nt} | R(x) \in V_i\}, \ (i = 1, 2).$$

We claim that (V_1', V_2') is a (td_1, td_2)-regular partition of Q_{nt}. This is because for any vertex x in V_1', $R(x)$ is in V_1. So $R(x)$ has d_1 neighbors in V_2, and each of which corresponds t neighbors of x in V_2', thus in total td_1 neighbors in V_2'. It is the same for vertices in V_2'.

By Proposition 1-3 we only need to consider the existence of (d_1, d_2)-regular partition of Q_n where $\gcd(d_1, d_2) = 1$ and $d_1 + d_2 = 2^s$ (where $s \leq n$), or equivalently, the existence of $(d, 2^s - d)$-regular partition of Q_n, where $s \leq n$ and d is odd. The following Lemma from [2] showed that when $n = 2^s - 1$ such regular partition always exists.

Lemma 1. *[2] There exists a $(t, 2^s - t)$-regular partition of Q_{2^s-1}, for any integer s, t with $0 < t < 2^s$.*

The following theorem shows how to construct the $(t, 2^s - t)$-regular partition for $n = 2^s - r$ (where $r \leq t$).

Theorem 2. *Suppose there exists a $(t, 2^s - t)$-regular partition for Q_{2^s-r} and $t > r$, then there exists a $(t, 2^{s+1} - t)$-regular partition for $Q_{2^{s+1}-r-1}$.*

Proof. For convenience, let $m = 2^s - r$. Suppose that (V_1, V_2) is a $(t, 2^s - t)$-regular partition for Q_m. Observer that $2^{s+1} - (r+1) = 2m+r-1$. We want to construct a $(t, 2^{s+1} - t)$-regular partition for Q_{2m+r-1}. The basic idea of the construction is as follows:

We start from set V_2. We construct a collection of linear equation systems, each of which corresponds to a node in V_2. The variables of the linear systems are the $(2m + r - 1)$ bits of node $x \in Q_{2m+r-1}$. Let V_2' be the union of solutions of these linear equation systems, and V_1' be the complement of V_2'. Then (V_1', V_2') is the $(t, 2^{s+1} - t)$-regular partition as we desired.

Here is the construction. Since (V_1, V_2) is a $(t, 2^s - t)$ regular partition for Q_m, the subgraph induced by V_2 of Q_m is a $(t-r)$-regular bipartite graph, i.e. for every node $p \in V_2$, there are $(t-r)$ neighbors of p in V_2. By Hall's theorem [9], we can find a perfect matching of nodes in V_2. For each $p \in V_2$, suppose q is the corresponding node of p in the matching. Let $I(p)$ be the index of bit at which p and q are different. Note that in this case $I(p) = I(q)$.

Now for each node $p = (p_1, \ldots, p_m) \in V_2$, we construct a linear equation system as follows:

$$
\begin{cases}
x_1 + x_2 = p_1, \\
x_3 + x_4 = p_2, \\
\cdots \quad \cdots \quad \cdots, \\
x_{2m-1} + x_{2m} = p_m, \\
\sum_{j=1}^{m} x_{2j-1} + x_{2I(p)} + \sum_{j=1}^{r-1} x_{2m+j} = 0.
\end{cases}
\tag{1}
$$

Note that all variables x_i and the addition operations are over $\mathbb{F}_2$. Denote by $S(p) \subseteq Q_{2m+r-1}$ the solutions of this linear system. For convenience, let $f : Q_{2m+r-1} \to Q_m$ be the operator such that

$$f(x_1, \ldots, x_{2m+r-1}) = (x_1 + x_2, x_3 + x_4, \ldots, x_{2m-1} + x_{2m}).$$

Then in the linear system (1) the first m equations is nothing but $f(x) = p$.

Let $V_2' = \bigcup_{p \in V_2} S(p)$, and $V_1' = Q_{2m+r-1} \setminus V_2'$ be its complement. We claim that (V_1', V_2') is a $(t, 2^{s+1} - t)$-regular partition of Q_{2m+r-1}.

To begin with, observe the following two facts.

Observation 1 For every $x \in V_2'$, we have $f(x) \in V_2$. It can be seen clearly from the first m equations in each equation system.

Observation 2 If $k \le 2m$, then $f(x^{(2k)}) = f(x^{(2k-1)}) = (f(x))^{(k)}$. If $k > 2m$, $f(x^{(k)}) = f(x)$. Recall the $x^{(i)}$ is the node obtained by flipping the ith bit of x. The observation can be seen from the definition of $f(x)$.

For any node $x \in V_1'$, we show that there are t different ways of flipping one bit of x so that we can get a node in V_2'. There are two possible cases:

Case 1: $f(x) \notin V_2$. In this case if we flip the ith bit of x for some $i > 2m$, then from Observation 2, $f(x^{(i)}) = f(x)$, so $f(x^{(i)})$ will remain not in V_2, and therefore $x^{(i)}$ will not be in V_2', by Observation 1. So we can only flip the bit in $\{x_1, \ldots, x_{2m}\}$.

Suppose by flipping the ith bit of x we get $x^{(i)} \in V_2'$ ($i \in [2m]$), from the definition of V_2' we have : $f(x^{(i)}) \in V_2$, and $x^{(i)}$ satisfies the last equation in the equation systems corresponding to $f(x^{(i)})$:

$$\sum_{j=1}^{m} x_{2j-1}^{(i)} + x_{2I(f(x^{(i)}))}^{(i)} + \sum_{j=1}^{r-1} x_{2m+j}^{(i)} = 0. \tag{2}$$

Since $f(x) \notin V_2$ and (V_1, V_2) is a $(t, 2^s - t)$-regular partition of Q_m, so there are exactly t neighbors of $f(x)$ in V_2, which implies there are t bits of $f(x)$ we can flip to get a node in V_2. Let $\{j_1, \ldots, j_t\} \subseteq [m]$ be these bits, i.e. $f(x)^{(j_1)}, \ldots, f(x)^{(j_t)} \in V_2$, by Observation 2,

$$f(x^{(2j_k-1)}) = f(x^{(2j_k)}) = f(x)^{(j_k)} \in V_2, \quad (k = 1, \ldots, t).$$

But exactly one of $\{x^{(2j_k-1)}, x^{(2j_k)}\}$ satisfies the equation (2) (here we use the fact $f(x) \notin V_2$, then $j_k \ne I(f(x)^{(j_k)})$). Thus totally, there are t possible i's such that $x^{(i)} \in V_2'$.

Case 2: $f(x) \in V_2$. Since $x \notin V_2'$, the last linear equation must be violated, i.e.

$$\sum_{j=1}^{m} x_{2j-1} + x_{2I(f(x))} + \sum_{j=1}^{r-1} x_{2m+j} = 1. \tag{3}$$

We further consider three cases here: flip a bit in $\{x_1, \ldots, x_{2m}\} \setminus \{x_{2I(f(x))}, x_{2I(f(x))-1}\}$; flip a bit in $\{x_{2I(f(x))}, x_{2I(f(x))-1}\}$; flip a bit in $\{x_{2m+1}, \ldots, x_{2m+r-1}\}$:

a) if $i \in [m]$, $i \ne I(f(x))$. If $f(x)^{(i)} \notin V_2$, both $x^{(2i)}$ and $x^{(2i-1)}$ will be in V_1' since first m equations cannot be satisfied. Otherwise, since (V_1, V_2) is a $(t, 2^s - t)$-regular partition of Q_m, there are $m - (2^s - t) - 1 = (2^s - r) - (2^s - t) - 1 = t - r - 1$ such index i. Note that for all such i's, exactly one of $x^{(2i-1)}$ and $x^{(2i)}$ is in V_2', depending on the value of Equation 3 for $f(x)^{(i)}$. $x^{(2i)} \in V_2'$ if it is originally 0, $x^{(2i-1)} \in V_2'$ if it is originally 1. This is based on the fact that $I(f(x)^{(i)}) \ne i$. Thus in this case there are $(t - r - 1)$ neighbors of x in V_2'.

b) if $i = I(f(x))$, then both of $x^{(2i-1)}, x^{(2i)}$ are in V_2', since $I(f(x)^{(i)}) = i$ in this case. There are 2 such neighbors.

c) if $i > 2m$, then every $x^{(i)}$ is in V_2', there are $r - 1$ such neighbors.

Hence, totally x has $(t - r - 1) + 2 + r - 1 = t$ neighbors in V_2'.

The rest thing is to show that every node $x \in V_2'$ has $(2^{s+1} - t)$ neighbors in V_1'. The proof is similar to the proof of Case 2 above, we consider three cases:

a) If $i \in [m]$, $i \ne I(f(x))$, and $f(x)^{(i)} \in V_2$. Then exactly one of $x^{(2k-1)}, x^{(2k)}$ in V_2', thus there are $m - (2^s - t) - 1 = 2^s - r - (2^s - t) - 1 = t - r - 1$ such neighbors of x in V_2'.

b) If $i = I(f(x))$ both $x^{(2i-1)}, x^{(2i)}$ are not in V_2'.

c) If $i > 2m$, then every $x^{(i)}$ is not in V_2'.

Hence totally, x has $(t-r-1)$ neighbors in V_2', and therefore $(2m+r-1)-(t-r-1) = 2^{s+1}-t$ neighbors in V_1'.

Hence we prove that (V_1', V_2') is indeed a $(t, 2^{s+1} - t)$-regular partition of $Q_{2^{s+1}-r-1}$.

Theorem 3. *For any odd number t and any $0 < c \leq t$, when $s \geq \lceil \log t \rceil + c - 1$, there exists a $(t, 2^s - t)$-regular partition of Q_{2^s-c}.*

Proof. Let $s_0 = s - c + 1$. Note that we have $2^{s_0} > t > 0$, by Lemma 1, there exists a $(t, 2^{s_0} - t)$-regular partition of $Q_{2^{s_0}-1}$. By repeatedly using Theorem 2, we obtain that there exists $(t, 2^{s_0+1} - t)$-regular partition of $Q_{2^{s_0+1}-2}$, $(t, 2^{s_0+2} - t)$-regular partition of $Q_{2^{s_0+2}-3}$. Finally we get a $(t, 2^{s_0+c-1} - t)$-regular partition of $Q_{2^{s_0+c-1}-c}$, which is just a $(t, 2^s - t)$-regular partition of Q_{2^s-c}.

Combining Proposition 2, Proposition 3 and Theorem 3, we have the following corollary.

Corollary 1. *Suppose $d_1 = dt, d_2 = d(2^s - t), n = d(2^s - c)$, where d, t, s are positive integers with $0 < t < 2^s$, $c \leq t$ and $s \geq \lceil \log t \rceil + c - 1$, then there exists a (d_1, d_2)-regular partition for Q_n.*

4   The Maximum Winning Probability $P_{n,k}$

The following lemma characterizes the relationship between the maximum winner probability $P_{n,k}$ and the minimum k-dominating set of Q_n. The same result was showed in [6] for $k = 1$.

Lemma 2. *Suppose D is a k-dominating set of Q_n with minimum number of vertices. Then*

$$P_{n,k} = 1 - \frac{|D|}{2^n}.$$

Proof. Given a k-dominating set D of Q_n, the following strategy will have winning probability at least $1 - \frac{|D|}{2^n}$: For any certain placement of hats, each player can see all hats but his own, so player p_i knows that current placement h is one of two adjacent nodes $\{x, x^{(i)}\}$ of Q_n. If $x \in D$ (or $x^{(i)} \in D$), he guesses that the current placement is $x^{(i)}$ (or x), otherwise he passes. We claim that by using this strategy, players win the game when the placement is a node which is not in D. Observe that since D is a k-dominating set, for any node $y \notin D$, y has l neighbors $y^{(i_1)}, y^{(i_2)}, \ldots, y^{(i_l)}$ that are in D, where $l \geq k$. According to the strategy desribed, players $p_{i_1}, \ldots, p_{i_l}$ would guess correctly and all other players will pass. This shows the winning probability is at least $1 - \frac{|D|}{2^n}$.

Next we show that $P_{n,k} \leq 1 - \frac{|D|}{2^n}$. Suppose we have a strategy with winning probability $P_{n,k}$. We prove that there exists a k-dominating set D_0, such that $|D_0| = 2^n(1 - P_{n,k})$. The construction is straightforward: Let $D_0 = \{h \in Q_n : h$ is not a winning placement$\}$. Thus $|D_0| = 2^n(1 - P_{n,k})$. For every winning placement $h \notin D_0$, suppose players $p_{i_1}, \ldots, p_{i_l}$ will guess correctly $(l \geq k)$, consider the placement $h^{(i_1)}$, which differs from h only at player p_{i_1}'s hat, so player p_{i_1} will guess incorrectly in this case, thus $h^{(i_1)} \in D_0$. Similarly $h^{(i_2)}, \ldots, h^{(i_l)} \in D_0$, therefore D_0 is a k-dominating set. We have

$$|D| \leq |D_0| = 2^n(1 - P_{n,k}),$$

which implies

$$P_{n,k} \leq 1 - \frac{|D|}{2^n}.$$

Combining these two results, we have $P_{n,k} = 1 - \frac{|D|}{2^n}$ as desired.

Proposition 4. *The following properties hold:*

(a) If $n_1 < n_2$ then $P_{n_1,k} \le P_{n_2,k}$.
(b) (n, k) is perfect iff there exists a (k, n)-regular partition of Q_n.
(c) For any $t \in \mathbb{N}$, $P_{nt,kt} \ge P_{n,k}$. As a consequence, if (n, k) is perfect, (nt, kt) is perfect.

Proof. For part (a), suppose that D is a minimum k-dominating set of Q_{n_1}. We make $2^{n_2-n_1}$ copies of Q_{n_1}, and by combining them we get a Q_{n_2}, which has dominating set of size $2^{n_2-n_1}|D|$. By Lemma 2, $P_{n_2,k} \ge 1 - \frac{2^{n_2-n_1}|D|}{2^{n_2}} = P_{n_1,k}$.

For part (b), suppose (U, V) is a (k, n)-regular partition of Q_n, note that V is a k-dominating set of Q_n and $|V| = \frac{k}{n+k} \cdot 2^n$, thus V is a minimum k-dominating set of Q_n. We have that $P_{n,k} = 1 - \frac{|V|}{2^n} = \frac{n}{n+k}$, which implies that (n, k) is perfect.

On the other hand, if (n, k) is perfect, suppose D is the minimum k-dominating set, we have $|D| = \frac{k}{n+k} \cdot 2^n$. It can be observed that $(Q_n \setminus D, D)$ is a (k, n)-regular partition of Q_n.

For part (c), since $\frac{n}{n+k} = \frac{nt}{nt+kt}$, once $P_{nt,kt} \ge P_{n,k}$ holds, it's an immediate consequence that the perfectness of (n, k) implies the perfectness of (nk, nt).

Suppose for n players, we have a strategy $\mathcal{S}$ with probability of winning $P_{n,k}$. For nt players, we divide them into n groups, each of which has t players. Each placement $h = (h_1, h_2, \ldots, h_{nt})$ of nt players can be mapped to a placement $P(h)$ of n players in the following way: for Group i, suppose the sum of colors in the group is w_i, i.e.

$$w_i(h) = \sum_{j=ti-t+1}^{ti} h_j, \quad (1 \le i \le n).$$

Let $P(h) = (w_1(h), w_2(h), \ldots, w_n(h))$ be a placement of n players. Each player in Group i knows the color of all players in $P(h)$ other than Player i, thus he uses Player i's strategy s_i in $\mathcal{S}$ to guess the sum of colors in Group i or passes. Moreover once he knows the sum, his color can be uniquely determined.

Note that the players in Group i would guess correctly or incorrectly or pass, if and only if Player i in the n-player-game would do. Since the hat placement is uniformly at random, the probability of winning using this strategy is at least $P_{n,k}$, thus $P_{nt,kt} \ge P_{n,k}$.

Now we can prove our main theorem:

Theorem 1 *For any $d, k, s \in \mathbb{N}$ with $s \ge \lceil \log k \rceil + k - 1$, $(d(2^s - k), dk)$ is perfect, in particular, $(2^s - k, k)$ is perfect.*

Proof. It is an immediate consequence of part (b) of Proposition 4 and Corollary 1.

Remark: By Proposition 1 and Proposition 4(b) there is a simple necessary condition for (n, k) to be perfect, $n + k = \gcd(n, k)2^t$. Theorem 1 indicates that when $n + k = \gcd(n, k)2^t$ and n is sufficiently large, (n, k) is perfect. The necessary condition and sufficient condition nearly match in the sense that for each k, there's only a few n that we don't know whether (n, k) is perfect. Moreover, the following proposition shows that the simple necessary condition can't be sufficient. The first counterexample is $(5, 3)$, it is not perfect while it satisfies the simple necessary condition. But $(13, 3)$ is perfect by Theorem 1 and more generally for all $s \ge 4$, $(2^s - 3, 3)$ is perfect. We verified by computer program that $(2^4 - 5, 5) = (11, 5)$ is not perfect, while our main theorem implies that $(2^7 - 5, 5) = (123, 5)$ is perfect. By applying Theorem 2 in a more delicate way, we

could obtain $(59, 5)$ is perfect as well. [1] But we still don't know whether the case between them, $(2^5 - 5, 5) = (27, 5)$, is perfect.

Proposition 5. (n, k) *is not perfect unless* $2k + 1 \leq n$ *when* $n \geq 2$ *and* $k < n$.

Proof. Suppose (n, k) is perfect. According to part (b) of Proposition 4, we can find (U, V), a (k, n)-regular partition of Q_n. Suppose x is some node in U, and y is some neighbor of x which is also in U, y has k neighbors in V. They all differ from x at exactly 2 bits and one of them is what y differs from x at, i.e. each of them "dominates" 2 neighbors of x, one of them is y. So x has totally $k + 1$ neighbors "dominated" by k of nodes in V. Since all nodes in V are pairwise nonadjacent, these $k + 1$ nodes must be in U. Now we have $k + 1$ neighbors of x are in U and k neighbors are in V, it has totally n neighbors. We must have $2k + 1 \leq n$.

For each odd number k, let $s(k)$ be the smallest number such that $(2^{s(k)} - k, k)$ is perfect. We know that $s(k) \in [\lceil \log k \rceil, \lceil \log k \rceil + k - 1]$. The following proposition indicates that all $s \geq s(k)$, $(2^s - k, k)$ is also perfect.

Proposition 6. *If* $(2^s - k, k)$ *is perfect,* $(2^{s+1} - k, k)$ *is perfect.*

Proof. If $(2^s - k, k)$ is perfect, by Proposition 4(b) there is a $(k, 2^s - k)$-regular partition of Q_{2^s-k}. Thus by Proposition 2, we have a $(k, 2^s - k)$-regular partition of Q_{2^s-k+1}. Combine this partition and Theorem 2, we get a $(k, 2^{s+1} - k)$-regular partition of $Q_{2^{s+1}-k}$. Therefore $(2^{s+1} - k, k)$ is perfect.

Using Theorem 1 we can give a general lower bound for the winning probability $P_{n,k}$. Recall that there's upper bound $P_{n,k} \leq 1 - \frac{k}{n+k}$.

Lemma 3. $P_{n,k} > 1 - \frac{2k}{n+k}$, *when* $n \geq 2^{\lceil \log k \rceil + k - 1} - k$.

Proof. Let n' be the largest integer of form $2^t - k$ which is no more than n. By Theorem 1, (n', k) is perfect, i.e. $P_{n',k} = 1 - \frac{k}{n'+k}$. By part (a) of Proposition 4, $P_{n,k} \geq P_{n',k}$. On the other hand we have $n + k < 2^{t+1}$, so we have

$$P_{n,k} \geq 1 - \frac{k}{n' + k} = 1 - \frac{2k}{2^{t+1}} > 1 - \frac{2k}{n + k}.$$

Corollary 2. *For any integer* $k > 0$, $\lim_{n \to \infty} P_{n,k} = 1$.

5 Conclusion

In this paper we investigated the existence of regular partition for boolean hypercube, and its applications in finding perfect strategies of a new hat guessing games. We showed a sufficient condition for (n, k) to be perfect, which nearly matches the necessary condition. Several problems remain open: for example, determine the minimum value of $s(k)$ such that $(2^{s(k)} - k, k)$ is perfect, and determine the exact value of $P_{n,k}$. It is also very interesting to consider the case when there are more than two colors in the game.

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[1] By Theorem 2, a $(3, 1)$-regular parition of Q_3 implies a $(3, 5)$-regular parition of Q_6, by switching the two parts of which, we have a $(5, 3)$-regular partition of Q_6. Then by applying Theorem 2 three times, we have $(5, 11)$-regular partition of Q_{12}, and then $(5, 27)$-partition of Q_{28}, and then $(5, 59)$-partition of Q_{59}, which implies that $(5, 59)$ is perfect by Proposition 4 (b).

6   Appendix

Here are some small values of n and k satisfying the necessary condition and being coprime, $n = 2^s - k$ for some s. "Y" means we know it is perfect, "N" means we know it is not perfect, and "U" means we are unsure.

k \ s	1	2	3	4	5	6	7	8	9
1	Y	Y	Y	Y	Y	Y	Y	Y	Y
3	-	-	N	Y	Y	Y	Y	Y	Y
5	-	-	-	N	U	Y	Y	Y	Y
7	-	-	-	N	U	U	U	U	Y
9	-	-	-	-	U	U	U	U	U
11	-	-	-	-	N	U	U	U	U

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