Lecture 23 More *NP*-Complete Problems

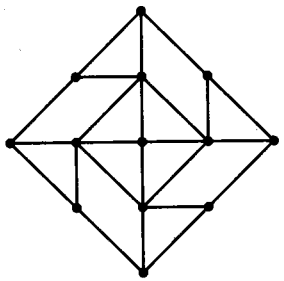
23.1 Planar Graph Colorability

Often in problems with a parameter k like k-CNFSat and k-colorability, larger values of k make the problem harder. This is not always the case. Consider the problem of determining whether a *planar* graph has a k-coloring. The problem is trivial for $k = 1$, easy for $k = 2$ (check by DFS or BFS whether the graph is bipartite, *i.e.* has no odd cycles), and trivial for $k = 4$ or greater by the Four Color Theorem, which says that every planar graph is 4-colorable. This leaves $k = 3$. We show below that 3-colorability of planar graphs is no easier than 3-colorability of arbitrary graphs. This result is due to Garey, Johnson, and Stockmeyer [40]; see also Lichtenstein [72] for some other *NP*-completeness results involving planar graphs.

We will reduce 3-colorability of an arbitrary graph to the planar case. Given an undirected graph $G = (V, E)$, possibly nonplanar, embed the graph in the plane arbitrarily, letting edges cross if necessary. We will replace each

edge crossing with the planar widget W shown below.

W =

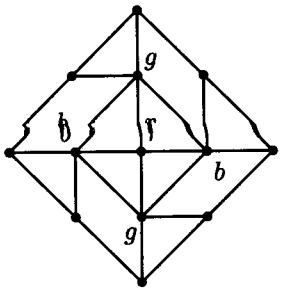


The widget W is a planar graph with the following interesting properties:

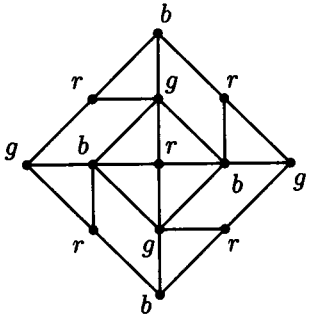
(i) in any legal 3-coloring of W, the opposite corners are forced to have the same color;

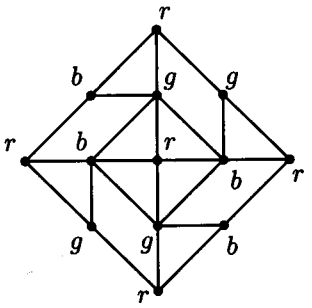
(ii) any assignment of colors to the corners such that opposite corners have the same color extends to a 3-coloring of all of W.

To see this, color the center of W red; then the vertices adjacent to the center must be colored blue or green alternately around the center, say



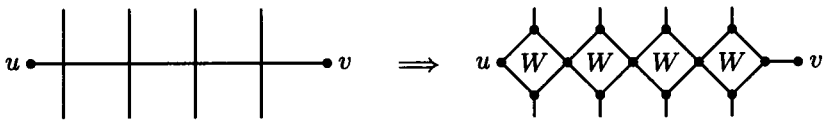
Now the northeast vertex can be colored either red or green. In either case, the colors of all the remaining vertices are forced (proceed counterclockwise to obtain the left hand coloring and clockwise to obtain the right hand coloring):





All other colorings are obtained from these by permuting the colors.

For each edge (u, v) in E, replace each point at which another edge crosses (u, v) in the embedding with a copy of W. Identify the adjacent corners of these copies of W and identify the outer corners of the extremal copies with u and v, all except for one pair, which are connected by an edge. The following diagram illustrates an edge (u, v) with four crossings before and after this operation. In this diagram, the copy of W closest to v is connected to v by an edge, and all other adjacent corners of copies of W are identified.



The resulting graph $G' = (V', E')$ is planar. If

$$\chi : V' \rightarrow \{\text{red, blue, green}\}$$

is a 3-coloring of G', then property (i) of W implies that χ restricted to V is a 3-coloring of G. Conversely, if $\chi : V \rightarrow \{\text{red, blue, green}\}$ is a 3-coloring of G, then property (ii) of W allows χ to be extended to a 3-coloring of G'.

We have given a reduction of the 3-colorability problem for an arbitrary graph to the same problem restricted to planar graphs. Thus the latter problem is as hard as the former.

23.2 *NP*-Completeness

The following definitions lay the foundations of the theory of *NP*-completeness. More detail can be found in [3, 39].

We fix once and for all a finite alphabet Σ consisting of at least two symbols. From now on, we take Σ^* to be the problem domain, and assume that instances of decision problems are encoded as strings in Σ^* in some reasonable way.

Definition 23.1 The complexity class *NP* consists of all decision problems $A \subseteq \Sigma^*$ such that A is the set of input strings accepted by some polynomial-time-bounded nondeterministic Turing machine. The complexity class P consists of all decision problems $A \subseteq \Sigma^*$ such that A is the set of input strings accepted by some polynomial-time-bounded deterministic Turing machine. □

Note that $P \subseteq NP$ since every deterministic machine is a nondeterministic one that does not happen to make any choices. It is not known whether $P = NP$; this is arguably the most important outstanding open problem in computer science.

Definition 23.2 The set A is *NP-hard (with respect to the reducibility relation* $\leq_m^P$*)* if $B \leq_m^P A$ for all $B \in NP$. □

Theorem 23.3 *If A is NP-hard and $A \in P$, then $P = NP$.*

Proof. For any $B \in NP$, compose the polynomial-time algorithm for A with the polynomial-time function reducing B to A to get a polynomial time algorithm for B. □

Definition 23.4 The set A is *NP-complete* if A is *NP*-hard and $A \in NP$. □

Theorem 23.5 *If A is NP-complete, then*

$$A \in P \leftrightarrow P = NP .$$

Definition 23.6 The complexity class *coNP* is the class of sets $A \subseteq \Sigma^*$ whose complements $\overline{A} = \Sigma^* - A$ are in *NP*. A set B is *coNP-hard* if every problem in *coNP* reduces in polynomial time to B. It is *coNP-complete* if in addition it is in *coNP*. □

The following theorem is immediate from the definitions.

Theorem 23.7

1. *$A \leq_m^P B$ iff $\overline{A} \leq_m^P \overline{B}$.*
2. *A is NP-hard iff $\overline{A}$ is coNP-hard.*
3. *A is NP-complete iff $\overline{A}$ is coNP-complete.*
4. *If A is NP-complete then $A \in coNP$ iff $NP = coNP$.*

It is unknown whether $NP = coNP$.

We will show later that the problems CNFSat, 3CNFSat, Clique, Vertex Cover, and Independent Set, which we have shown to be $\equiv_m^P$-equivalent, are all in fact *NP*-complete.

23.3 More *NP*-complete problems

Before we prove the *NP*-completeness of the problems we have been considering, let us consider some more problems in this class. Some of these problems, such as Traveling Salesman, Bin Packing, and Integer Programming, are very natural and important in operations research and industrial engineering. We start with the *exact cover problem.*

Definition 23.8 (Exact Cover) Given a finite set X and a family of subsets S of X, is there a subset $S' \subseteq S$ such that every element of X lies in exactly one element of S'? □

We show that the problem Exact Cover is *NP*-hard by reduction from the problem of 3-colorability of undirected graphs. See [39] for a different approach involving the 3-dimensional matching problem.

Lemma 23.9 *3-Colorability* $\leq_m^P$ *Exact Cover.*

Proof. Suppose we are given an undirected graph $G = (V, E)$. We show how to produce an instance (X, S) of the exact cover problem for which an exact cover exists iff G has a 3-coloring.

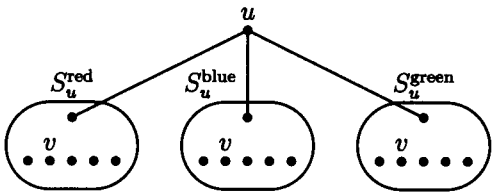
Let $C = \{\text{red, blue, green}\}$. For each $u \in V$, let $N(u)$ be the set of neighbors of u in G. Since G is undirected, $u \in N(v)$ iff $v \in N(u)$.

For each $u \in V$, we include u in X along with $3(|N(u)| + 1)$ additional elements of X. These $3(|N(u)| + 1)$ additional elements are arranged in three disjoint sets of $|N(u)| + 1$ elements each, one set corresponding to each color. Call these three sets S_u^{red}, S_u^{blue}, S_u^{green}. For each color $c \in C$, pick a special element p_u^c from S_u^c and associate the remaining $|N(u)|$ elements of S_u^c with the elements of $N(u)$ in a one-to-one fashion. Let q_{uv}^c denote the element of S_u^c associated with $v \in N(u)$.

The set S will contain all two element sets of the form

$$\{u, p_u^c\} \tag{31}$$

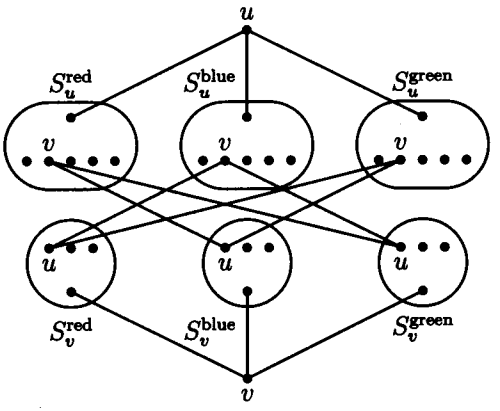
for $u \in V$ and $c \in C$, as well as all the sets S_u^c for $u \in V$ and $c \in C$. Here is a picture of what we have so far for a vertex u of degree 5 with $v \in N(u)$. The ovals represent the three sets S_u^c and the lines represent the three two-element sets (31).



To complete S, we include all two element sets of the form

$$\{q_{uv}^c, q_{vu}^{c'}\} \tag{32}$$

for all $(u, v) \in E$ and $c, c' \in C$ with $c \neq c'$. Here is a picture showing a part of the construction for two vertices u and v of degrees 5 and 3 respectively, where (u, v) in E. The six lines in the center represent the two-element sets (32).



We now argue that the instance (X, S) of Exact Cover just constructed is a "yes" instance, *i.e.* an exact cover $S' \subseteq S$ of X exists, iff the graph G has a 3-coloring. Suppose first that G has a 3-coloring $\chi : V \to C$. We construct an exact cover $S' \subseteq S$ as follows. For each vertex u, let S' contain the sets $\{u, p_u^{\chi(u)}\}$ and S_u^c for $c \neq \chi(u)$. This covers everything except points of the form $q_{uv}^{\chi(u)}$, where $(u, v) \in E$. For each edge (u, v), let S' also contain the set $\{q_{uv}^{\chi(u)}, q_{vu}^{\chi(v)}\}$. This set is in S since $\chi(u) \neq \chi(v)$. This covers all the remaining points, and each point is covered by exactly one set in S'.

Conversely, suppose S' is an exact cover. Each u is covered by exactly one set in S', and it must be of the form $\{u, p_u^c\}$ for some c. Let $\chi(u)$ be that c; we claim that χ is a valid coloring, *i.e.* that if $(u, v) \in E$ then $\chi(u) \neq \chi(v)$. For each u, since $\{u, p_u^{\chi(u)}\} \in S'$, we cannot cover p_u^c for $c \neq \chi(u)$ by any set of the form (31), since u is already covered; therefore they must be covered by the sets S_u^c, which are the only other sets containing the points p_u^c. The sets $\{u, p_u^{\chi(u)}\}$ and S_u^c, $c \neq \chi(u)$ cover all points except those of the form $q_{uv}^{\chi(u)}$, $(u, v) \in E$. The only way S' can cover these remaining points is by the sets (32). By construction of S, these sets are of the form $\{q_{uv}^{\chi(u)}, q_{vu}^{\chi(v)}\}$ for $(u, v) \in E$ and $\chi(u) \neq \chi(v)$. □