



# Kinematics Equations for Differential Drive and Articulated Steering

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## Introduction

A mobile robot, or vehicle, has 6 degrees of freedom (DOF) expressed by the *pose*:  $(x, y, z, Roll, Pitch, Yaw)$ . It is composed of two parts: the *position*  $= (x,y,z)$  and the *attitude*  $= (Roll, Pitch, Yaw)$ . Informally, *Roll* can be said to be to the sidewise rotation and *Pitch* the rotation forward or backwards. *Yaw*, commonly also denoted *Heading* or *Orientation*, refers to the direction in which the robot moves in the x-y plane.

For a robot on a two dimensional surface, the 2D pose  $(x,y,\theta)$ , where  $\theta$  denotes the heading, is sufficient to describe its motion. It is normally defined in a global coordinate system as illustrated below. Note that  $\theta$  is NOT an angular polar coordinate for the position, instead it points in the forward direction of the robot.

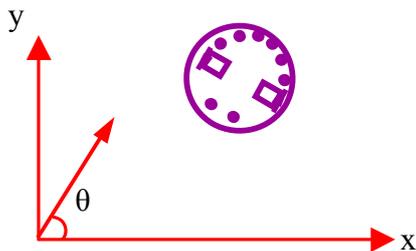


Figure 1. The robot's pose  $(x,y,\theta)$ , given in a global coordinate system

For a robot with *differential drive*, the direction of motion is controlled by separately controlling speeds  $v_l$  and  $v_r$  of the left and right wheels respectively. Many such robots have two wheels connected directly to motors, and in addition some kind of support wheel to keep the robot upright. Common examples of robot with differential drive are the Khepera robot and the Roomba vacuum-cleaning robot shown in Figure 2.

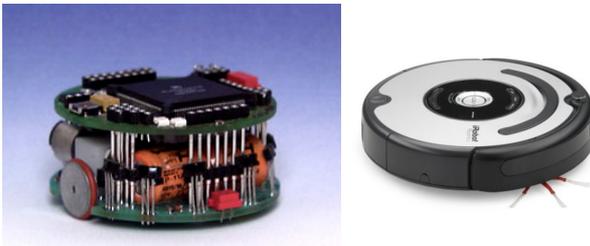


Figure 2. A Khepera robot and a Roomba vacuum-cleaning robot, both with two wheels and differential drive

The *forward kinematics equations* for a robot (or other vehicle) with differential drive are used to solve the following problem:

**Standing in the pose  $(x, y, \theta)$  at time  $t$ , determine the pose  $(x', y', \theta')$  at time  $t + \delta t$  given the control parameters  $v_l$  and  $v_r$ .**

The solution is typically used for automatic control of a robot such that it follows a wanted trajectory.

## Derivation of the forward kinematics equations

We start by looking at how a single rotating wheel moves. In Figure 3, a wheel, seen from above, is illustrated together with a local coordinate system. Motion along the  $Y$  axis is known as *roll*, everything else is known as *slip*. In the following we will assume that no slip occurs.

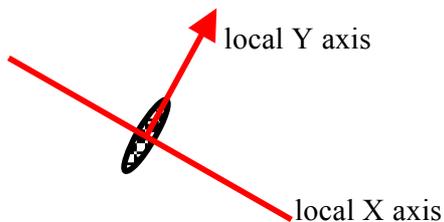


Figure 3. A single rotating wheel rolls along the local  $Y$  axis

For one full turn of the wheel, the center moves a distance  $2\pi r_w$  where  $r_w$  is the radius of the wheel. The implicit assumption is that no slip occurs and also that the motion is truly 2-dimensional. This means that the surface is flat and even.

For a robot with many rolling wheels, each wheel must roll along its own  $Y$  axis, and a common center point for rotation must exist, as illustrated in Figure 4. This point is called *ICC* (Instantaneous Center of Curvature) or *ICR* (Instantaneous Center of Rotation). The speed of each wheel has to be consistent with a rigid rotation of the vehicle in the sense that the wheels do not move relative to each other.

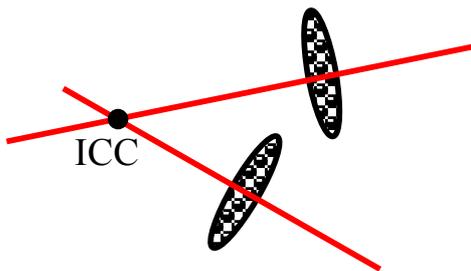


Figure 4. Two rotating wheels must share a common point of rotation

For a robot with differential drive, a pair of wheels is mounted on a common axis, see Figure 5. If the wheels are rotating on ground (i.e. there is no slipping), then there is a point ICC (provided  $v_r \neq v_l$ ) around which both wheels rotate. By varying  $v_r$  and  $v_l$ , ICC moves and different trajectories for the robot are chosen.

A central concept for the derivation of the kinematics equations is the *Angular velocity*  $\omega$  of the robot. It is defined as follows: Each wheel rotates around ICC along a circle with radius  $r$ .

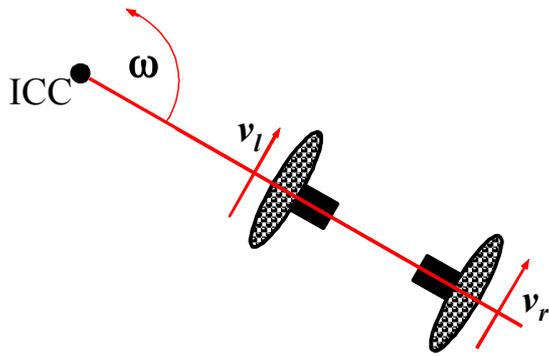


Figure 5. Wheel configuration for a robot with differential drive

The wheel speed  $v = 2\pi r / T$  where  $T$  is the time it would take to complete one full turn around ICC. The angular velocity  $\omega$  is defined as  $2\pi / T$  and typically has the unit radians (or degrees) per second. Combining the equations for  $v$  and  $\omega$  yields  $\omega = 2\pi / T = 2\pi r / rT = v/r$  and consequently

$$\omega r = v. \quad 1)$$

Note that plugging in  $r$  and  $v$  for both left and right wheel result in the same  $\omega$  (otherwise the wheels would move relative to each other). Hence, the following equations hold:

$$\omega (R+l/2) = v_r \quad 2)$$

$$\omega (R - l/2) = v_l \quad 3)$$

where  $R$  is the distance between ICC and the midpoint of the wheel axis, and  $l$  is the length of the wheel axis (see Figure 6). Solving for  $\omega$  and  $R$  yields

$$R = l/2(v_l + v_r) / (v_r - v_l) \quad 4)$$

$$\omega = (v_r - v_l) / l \quad 5)$$

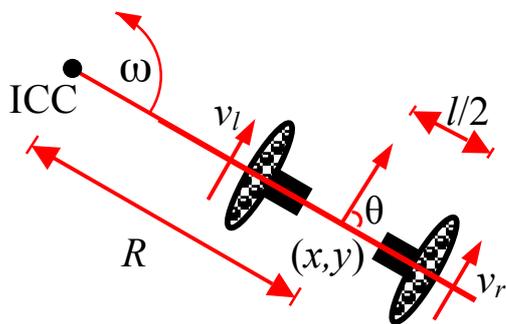


Figure 6. When left and right wheel rotate with different speeds, the robot rotates around a common point denoted ICC

These expressions for radius  $R$  and angular velocity  $\omega$  contain most necessary information to solve the forward kinematics problem. Assume that the robot rotates

around ICC with angular velocity  $\omega$  for  $\delta t$  seconds (see Figure 7). This will change the heading according to:

$$\theta' = \omega \delta t + \theta \quad (6)$$

where the center of rotation ICC is given by basic trigonometry as:

$$\text{ICC} = [\text{ICC}_x, \text{ICC}_y] = [x - R \sin \theta, y + R \cos \theta]. \quad (7)$$

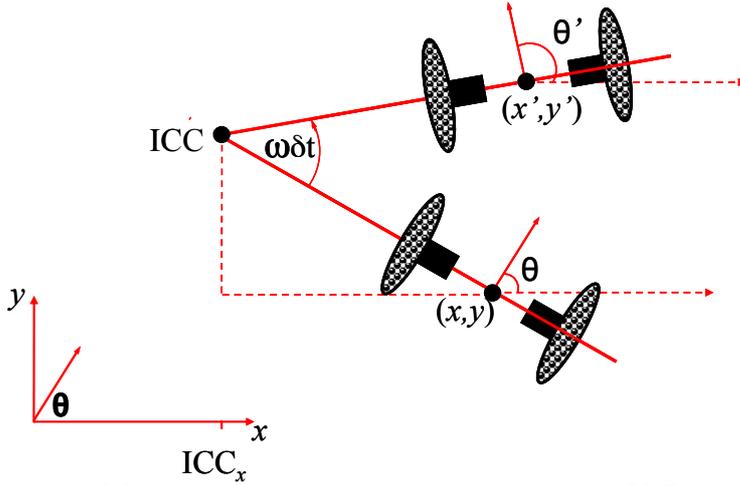


Figure 7. Rotating the robot  $\omega \delta t$  degrees around ICC

Given a starting position  $(x, y)$ , the new position  $(x', y')$  can be computed using a 2D rotational matrix. Rotation around ICC with angular velocity  $\omega$  for  $\delta t$  seconds yields the following position at time  $t + \delta t$ :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\omega \delta t) & -\sin(\omega \delta t) \\ \sin(\omega \delta t) & \cos(\omega \delta t) \end{bmatrix} \begin{bmatrix} x - \text{ICC}_x \\ y - \text{ICC}_y \end{bmatrix} + \begin{bmatrix} \text{ICC}_x \\ \text{ICC}_y \end{bmatrix}. \quad (8)$$

Hence, the new pose  $(x', y', \theta')$  can be computed from equations 6) and 8) given  $\omega$ ,  $\delta t$ ,  $R$ .  $\omega$  is given by Equation 5, but wheel speeds  $v_l$  and  $v_r$  are often hard to measure accurately. Instead, the rotation of each wheel can be measured, for instance by so called *wheel encoders*. These sensors are mounted on the wheel axes and deliver a binary signal for each *step* the wheels rotate (for an indoor robot, *step* is typically in the order of 0.1 mm). The signals are fed to digital counters such that  $v \delta t$ , the distances travelled from time  $t$  to  $t + \delta t$ , can be derived from the increase in counter value  $n$ :  $n \text{ step} = v \delta t$ . From this,  $v$  can be computed as:

$$v = n \text{ step} / \delta t. \quad (9)$$

Insertion in 3) and 4) yields:

$$R = l/2 (v_l + v_r) / (v_r - v_l) = l/2(n_l + n_r) / (n_r - n_l) \quad (10)$$

$$\omega \delta t = (v_r - v_l) \delta t / l = (n_r - n_l) \text{ step} / l \quad (11)$$

where  $n_l$  and  $n_r$ , and  $v_l$  and  $v_r$  are the encoder counts and speeds for left and right wheels respectively. Thus, if the robot is standing in pose  $(x,y,\theta)$  and moves  $n_l$  and  $n_r$  counts during a time step  $\delta t$ , the new pose  $(x',y',\theta')$  is given by

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} \cos(\omega\delta t) & -\sin(\omega\delta t) & 0 \\ \sin(\omega\delta t) & \cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x-ICC_x \\ y-ICC_y \\ \theta \end{pmatrix} + \begin{pmatrix} ICC_x \\ ICC_y \\ \omega\delta t \end{pmatrix} \quad 12)$$

where

$$R = l/2 (n_l + n_r) / (n_r - n_l) \quad 13)$$

$$\omega\delta t = (n_r - n_l) \text{ step} / l \quad 14)$$

$$ICC = [ x-R \sin\theta, y+R \cos\theta ]. \quad 15)$$

Note that equations 12)-15) are independent of  $\delta t$  (which is normally hard to estimate accurately).

The derived kinematics equations depend heavily on the design and geometry of the robot or vehicle. Different types of designs lead to entirely different equations. In Appendix 1, forward kinematics equations for a vehicle with articulated steering are derived.

## Inverse kinematics

While the forward kinematics equations provide an updated pose given certain wheel speeds (or encoder counts), we can also formulate an inverse problem:

**Standing in pose  $(x,y,\theta)$  at time  $t$ , determine control parameters  $v_l$  and  $v_r$  such that the pose at time  $t + \delta t$  is  $(x',y',\theta')$ .**

This problem most often has no solution, in the sense that the robot can not reach an arbitrary pose by simply setting appropriate values for wheels speeds  $v_l$  and  $v_r$  and let the motors run for a while. For some robots and vehicles it IS possible, and these vehicles are called *Holonomic*. However, most vehicles and robots are non-holonomic. For instance, a car is non-holonomic and this is why pocket parking is so hard.

For a non-holonomic robot, there are ways to increase the constrained mobility. If we allow a sequence of different  $(v_l, v_r)$ , there are normally infinitely many ways to move from one pose to another. We will study the special case with a robot controlled by a differential drive. By inserting values in Equations 12-15, we can identify two special cases of control:

1.

$v_r = v_l \Rightarrow n_r = n_l \Rightarrow R = \infty, \omega \delta t = 0 \Rightarrow$  The robot moves in a straight line and  $\theta$  remains the same.

2.

$v_r = -v_l \Rightarrow n_r = -n_l \Rightarrow R = 0, \omega \delta t = 2n_l \text{ step} / l$  and  $ICC = [ICC_x, ICC_y] = [x, y] \Rightarrow x' = x, y' = y, \theta' = \theta + \omega \delta t \Rightarrow$  The robot rotates in place about  $ICC$ . I.e.: any  $\theta$  is reachable while  $(x,y)$  is unchanged.

By combining these two operations, the following algorithm can be used to reach any target pose from any starting pose:

1. Rotate until the robot's orientation coincides with the line from the starting position to the target position:

$$v_r = -v_l = v_{rot}$$

2. Drive straight until the robot's position coincides with the target position:

$$v_r = v_l = v_{ahead}$$

3. Rotate until the robot's orientation coincides with the target orientation:

$$v_r = -v_l = v_{rot}$$

where  $v_{rot}$  and  $v_{ahead}$  can be chosen arbitrarily.

## Appendix 1: Forward kinematics equations for an articulated vehicle

In this section forward kinematics equations for an articulated vehicle are derived. Such a vehicle consists of two separate sections connected by an articulated joint. Steering is accomplished by controlling the angle of this joint. Articulated vehicles are most common in heavy working vehicles such as dumpers and forest machines (Figure 1).



Figure 1: Forest machine with articulated steering

For slip free motion, all wheels roll in full contact with the ground, in a direction perpendicular to its axis of rotation. The distance covered can be computed from the size and rotational speed of the wheel. For many reasons, this is an idealized situation. Slip is often significant and furthermore hard to model. For this reason, slip free motion is often assumed when solving the kind of problems stated above.

For a multi-wheel vehicle with no slip, the intersection of the wheel axes is the center point for rotation when the vehicle moves. This point is called ICC (*Instantaneous Center of Curvature*) or ICR (*Instantaneous Center of Rotation*). In many cases, totally slip free motion is not geometrically possible. The situation for an articulated vehicle with four wheel axes is illustrated in Figure 2.

The varying steering angle  $\phi$  makes it impossible to construct the vehicle such that the axes intersect in one point. To avoid modeling slip, a common approach is to assume two virtual axes located in between the real axes in the front and rear part of the vehicle. Another complication is the width of the wheels. The outer part of a wheel is bound to travel a longer distance than the inner part in all curves. Hence, they will slip. Furthermore, the speeds of all wheels have to be controlled such that slip free motion is at least approximately possible. The outermost wheels have to rotate faster than the innermost ones, and the rear wheels have to rotate slower than the front wheels (assuming a longer rear part as in Figure 2). Depending on the mechanical construction and the control system of the vehicle, this is a more or less valid assumption. The following kinematics equations are derived with all above mentioned idealized assumptions.

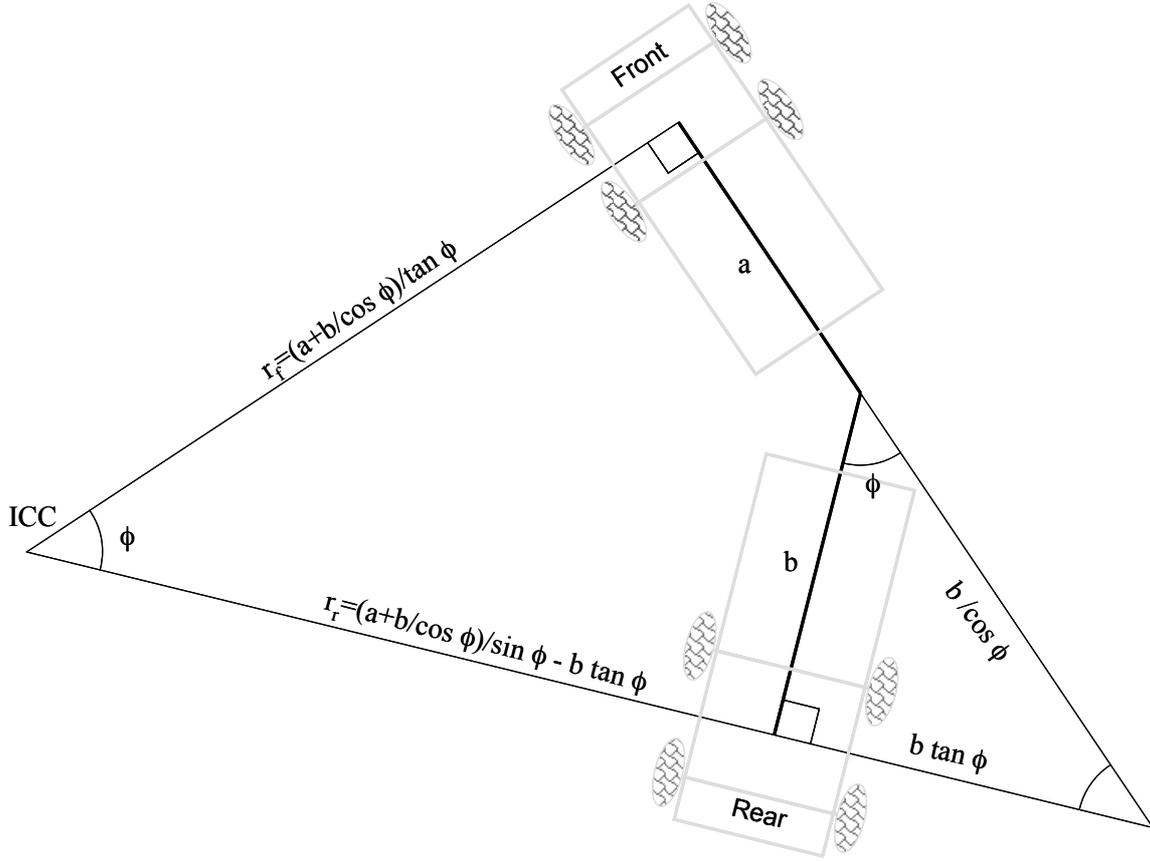


Figure 2. Derivation of radius of rotation for an articulated vehicle with steering angle  $\phi$ , front length  $a$  and rear length  $b$ .

The center of each (virtual) axis rotates around ICC along a circle with radius  $r$ . For an articulated vehicle,  $r$  is given by the geometry of the vehicle and the steering angle as illustrated in Figure 2. For the front axis, radius  $r_f$  is given by

$$r_f = (a+b/\cos\phi) / \tan\phi. \quad [1]$$

For the rear axis, radius  $r_r$  is given by

$$r_r = (a+b/\cos\phi) / \sin\phi - b \tan\phi. \quad [2]$$

Under the assumptions above, it suffices to study the motion of the front part of the vehicle, since the motion of the rear part is given by the geometry of the vehicle. Given a vehicle pose  $(x,y,\theta)$  measured at the middle of the virtual front axis at time  $t$ , coordinates  $(X_{ICC}, Y_{ICC})$  for ICC is given by

$$(X_{ICC}, Y_{ICC}) = (x-r \sin\theta, y+r \cos\theta), \quad [3]$$

where  $r$  for simplicity of notation denotes the radius  $r_f$ . A motion from pose  $(x,y,\theta)$  at time  $t$  to pose  $(x',y',\theta')$  at time  $t + \delta t$  is illustrated in Figure 3.

Since the vehicle moves along a circle, it will be useful to have an expression for the angular velocity  $\omega$  defined as  $2\pi / T$  (unit: radians/second), where  $T$  is the time it would take to complete one full turn around ICC.

The known vehicle speed  $v$  is assumed to be the speed at which the midpoint of the front axis moves (this is an additional assumption that may or may not be sufficiently valid). It can thus be expressed as  $2\pi r / T$ , which gives the following expression for  $\omega$ :

$$\omega = v/r. \quad [4]$$

The new heading  $\theta'$  at  $t + \delta t$  is given by

$$\theta' = \omega \delta t + \theta. \quad [5]$$

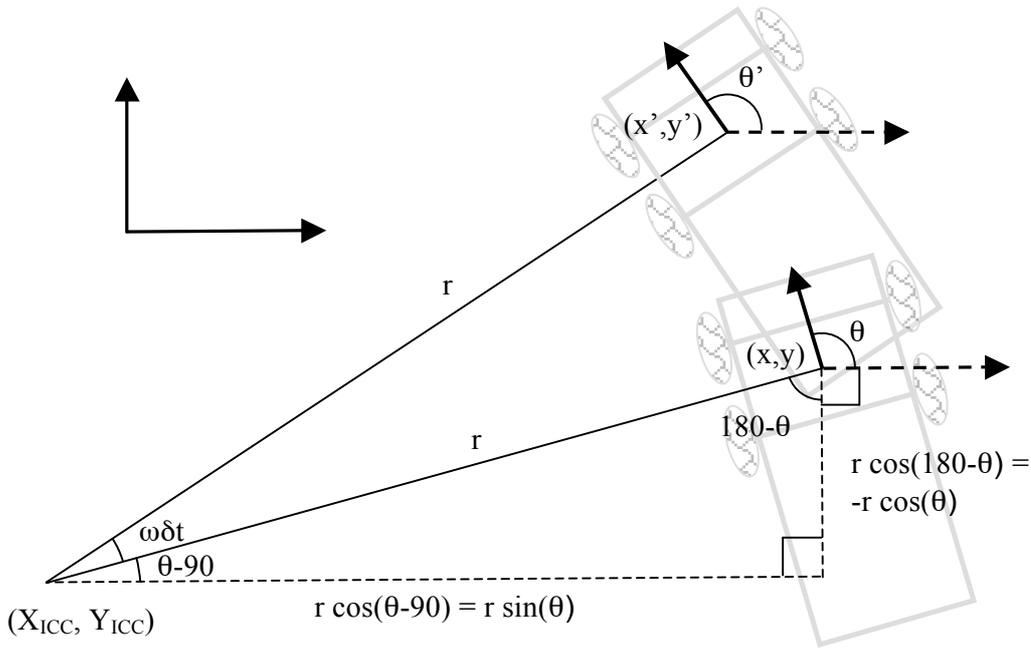


Figure 3. Rotation around ICC by an angle  $\omega \delta t$ . The vehicle pose changes from  $(x, y, \theta)$  to  $(x', y', \theta')$  expressed in a global coordinate system (top left). The coordinates of ICC are  $(x - r \sin \theta, y + r \cos \theta)$ .

The new position  $(x', y')$  at  $t + \delta t$  is computed by a 2D rotation of the point  $(x, y)$  by  $\omega \delta t$  degrees around the point ICC:

$$x' = \cos(\omega \delta t)(x - X_{ICC}) - \sin(\omega \delta t)(y - Y_{ICC}) + X_{ICC} \quad [6]$$

$$y' = \sin(\omega \delta t)(x - X_{ICC}) + \cos(\omega \delta t)(y - Y_{ICC}) + Y_{ICC}. \quad [7]$$

To summarize and return to the original problem for the articulated vehicle in Figure 2: Given an initial pose  $(x, y, \theta)$ , a reported vehicle speed  $v$  and a steering angle  $\phi$  at time  $t$ , the pose  $(x', y', \theta')$  at time  $t + \delta t$  can be estimated by the following algorithm:

1.  $r = (a + b/\cos \phi) / \tan \phi$
2.  $\omega = v/r$
3.  $[X_{ICC}, Y_{ICC}] = [x - r \sin \theta, y + r \cos \theta]$
4.  $\theta' = \omega \delta t + \theta$

5.  $x' = \cos(\omega\delta t)(x - X_{ICC}) - \sin(\omega\delta t)(y - Y_{ICC}) + X_{ICC}$
6.  $y' = \sin(\omega\delta t)(x - X_{ICC}) + \cos(\omega\delta t)(y - Y_{ICC}) + Y_{ICC}$

As mentioned above, the derivation of the new pose makes several assumptions:

1. Assuming slip free motion (ignoring geometrical impossibilities, tires with finite width, inconsistent front and rear wheel speed, and slippery ground conditions).
2. The derivation of the equations uses two virtual wheel axes located in between the real wheel axes.
3. The value  $v$  is assumed to be the speed of the front part of the vehicle. Often, the available speed value is the estimated vehicle speed, based on the engine speed and the transmission. This value is not necessarily the same as the speed of the front part.