Non-separating Spanning Trees and Out-Branchings in Digraphs of Independence Number 2

Jørgen Bang-Jensen, Stéphane Bessy, Anders Yeo

Non-separating spanning trees and out-branchings in digraphs of independence number 2

J. Bang-Jensen* S. Bessy† A. Yeo‡

July 7, 2020

Abstract

A subgraph $H = (V, E')$ of a graph $G = (V, E)$ is **non-separating** if $G \backslash E'$, that is, the graph obtained from G by deleting the edges in E', is connected. Analogously we say that a subdigraph $X = (V, A')$ of a digraph $D = (V, A)$ is non-separating if $D \setminus A'$ is strongly connected. We study non-separating spanning trees and out-branchings in digraphs of independence number 2. Our main results are that every 2-arc-strong digraph D of independence number $\alpha(D) = 2$ and minimum in-degree at least 5 and every 2-arc-strong oriented graph with $\alpha(D) = 2$ and minimum in-degree at least 3 has a non-separating out-branching and minimum in-degree 2 is not enough. We also prove a number of other results, including that every 2-arc-strong digraph D with $\alpha(D) \leq 2$ and at least 14 vertices has a non-separating spanning tree and that every graph G with $\delta(G) \geq 4$ and $\alpha(G) = 2$ has a non-separating hamiltonian path.

Keywords: non-separating branching; spanning trees; digraphs of independence number 2; strongly connected; hamiltonian path.

1 Introduction

An **out-tree** in a digraph $D = (V, A)$ is a connected subdigraph T_s^+ of D in which every vertex of $V(T_s^+)$, except one vertex s (called the **root**) has exactly one arc entering. This is equivalent to saying that s can reach every other vertex of $V(T_s^+)$ by a directed path using only arcs of T_s^+. An **out-branching** in a digraph $D = (V, A)$ is a spanning out-tree, that is, every vertex of V is in the tree. We use the notation B_s^+ for an out-branching rooted at the vertex s. An **in-branching**, B_t^-, rooted at the vertex t is defined analogously. The following classical result due to Edmonds and the algorithmic proof due to Lovász [14] implies that one can check the existence of k arc-disjoint out-branchings in polynomial time.

Theorem 1 (Edmonds)**.** *[12] Let $D = (V, A)$ be a digraph and let $s \in V$. Then D contains k arc-disjoint out-branchings, all rooted at s, if and only if there are k arc-disjoint (s, v)-paths in D for every $v \in V$.*

*Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark (email: jbj@imada.sdu.dk). Part of this work was done while the author was visiting LIRMM, Université de Montpellier as well as INRIA Sophia Antipolis. Hospitality and financial support by both is gratefully acknowledged. Ce travail a bénéficié d'une aide du gouvernement franais, gérée par l'Agence Nationale de la Recherche au titre du projet Investissements d'Avenir UCAJEDI portant la référence no ANR-15-IDEX-01.

†LIRMM, CNRS, Université de Montpellier, Montpellier, France (email:stephane.bessy@lirmm.fr), financial suports: PICS CNRS DISCO and ANR DIGRAPHS n.194718.

‡Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark and Department of Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa (email: yeo@imada.sdu.dk).

Deciding the existence of arc-disjoint in-and out-branchings is considerably more difficult as shown by the following result due to Thomassen (the theorem and its proof can be found in [1]).

Theorem 2. *It is NP-complete to decide whether a digraph contains arc-disjoint branchings* B_s^+, B_t^- *for given vertices* s, t.

It was shown in [6] that the problem remains NP-complete even for 2-arc-strong 2-regular digraphs.

Thomassen conjectured that sufficiently high arc-connectivity will guarantee the existence of arc-disjoint in- and out-branchings with prescribed roots. As defined in Section 2, $\lambda(D)$ denotes is the arc-connectivity of the digraph D.

Conjecture 3. [16] There exists a natural number K such that every digraph D with $\lambda(D) \geq K$ contains arc-disjoint branchings B_s^+, B_t^- for every choice of $s, t \in V$.

It was pointed out in [2] that Conjecture 3 is equivalent to the following (the same value of K would work for both conjectures).

Conjecture 4. There exists a natural number K such that every digraph D with $\lambda(D) \geq K$ contains an out-branching which is arc-disjoint from some in-branching.

In this paper we study digraphs of independence number 2. The structure of digraphs with independence number 2 is not well understood and there are numerous interesting open problems. For instance it is an open problem whether the existence of vertex disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path for $i = 1, 2$ can be checked in polynomial time (for a partial result see [11]). In the case where we want arc-disjoint paths, a polynomial algorithm was given in [13].

Very recently the following result which settles Conjectures 3 and 4 for digraphs of independence number 2 was obtained. The result is best possible in terms of the arc-connectivity as there are infinitely many strong digraphs with independence number 2 and arbitrarily high minimum in-and out-degrees that have no out-branching which is arc-disjoint from some in-branching [2].

Theorem 5. *[2] Every digraph* $D = (V, A)$ *with* $\alpha(D) = 2$ *and* $\lambda(D) \geq 2$ *contains arc-disjoint branchings* B_s^+, B_t^- *for some choice of* $s, t \in V$.

Conjecture 6. [2] Every 2-arc-strong digraph $D = (V, A)$ with $\alpha(D) = 2$ has a pair of arc-disjoint branchings B_s^+, B_s^- for every choice of $s \in V$.

Conjecture 7. [2] Every 3-arc-strong digraph $D = (V, A)$ with $\alpha(D) = 2$ has a pair of arc-disjoint branchings B_s^+, B_t^- for every choice of $s, t \in V$.

In the present paper we are interested in the existence an out-branching B^+ in a strongly connected digraph D of independence number 2 such that the digraph $D \setminus A(B^+)$ that we obtain by deleting all arcs of B^+ is still strongly connected. Clearly if D has such an out-branching, then it also has arc-disjoint in- and out-branchings B_s^+, B_s^- from some vertex s (namely the root of B^+). The main result of the paper is the following which, besides being of interest in connection with Conjecture 11 below, also provides support for Conjecture 6.

Theorem 8. *Let* D *be a 2-arc-strong digraph with* $\alpha(D) \leq 2$. *If either of the following statements hold then there exists an out-branching,* B^+, *in* D, *such that* $D \setminus A(B^+)$ *is strongly connected.*

(i): $\delta^-(D) \geq 5$*, or*

(ii): $\delta^-(D) \geq 3$ *and* D *is an oriented graph (no cycles of length 2).*

Theorem 8 (ii) is best possible in the sense that there exists a digraph $\tilde{D}$, with $\alpha(\tilde{D}) = 2$ and $\lambda(\tilde{D}) \geq 2$ (and therefore $\delta^-(\tilde{D}) \geq 2$) which does not contain a non-separating out-branching. See Figure 6 and Proposition 24.

In Section 2 we provide some preliminary results. In particular, we show in Proposition 12 that there are infinitely many 2-arc-strong digraphs with independence number 2 and high in- and out-degrees that do not have an arc-partition into two spanning strong subdigraphs, implying that we cannot replace B^+ in Theorem 8 by some spanning strong subdigraph. We also describe some structural results on semicomplete digraphs that will be used in later sections. Finally we prove a structural result on strong spanning subdigraphs with few arcs in digraphs with $\alpha(D) = 2 \leq \lambda(D)$.

In Section 3 we characterize semicomplete digraphs with non-separating out-branchings and prove a more general result which will be used in the proof of Theorem 8.

In Section 4 we prove Theorem 8 and in Section 5 we study non-separating spanning trees in digraphs of independence number 2. The main result here is Theorem 23 which says that every 2-arc-strong digraph D with $\alpha(D) = 2$ and $n \geq 14$ vertices has a non-separating spanning tree. We conjecture that already $n \geq 9$ is enough, which would be best possible, and prove this in the case when D has a hamiltonian cycle and no cycle of length 2. In Section 6 we construct an infinite family of 2-arc-strong digraphs with $\alpha = 2$ for which every hamiltonian path is separating and in Section 7 we show that for undirected graphs with independence number 2 a non-separating hamiltonian path always exists, provided the minimum degree is at least 4. Finally, in Section 8 we pose a number of open problems.

2 Terminology and Preliminaries

Terminology not defined here or above is consistent with [3]. Let $D = (V, A)$ be a digraph. The **underlying graph** of D is the graph $UG(D) = (V, E)$ where $uv \in E$ if and only if there is at least one arc between u and v in D. For a non-empty subset $X \subset V$ we denote by $d_D^+(X)$ (resp. $d_D^-(X)$) the number of arcs with tail (resp. head) in X and head (resp. tail) in $V \setminus X$. We call $d_D^+(X)$ (resp. $d_D^-(X)$) the **out-degree** (resp. **in-degree**) of the set X. Note that X may be just a vertex. We will drop the subscript when the digraph is clear from the context. We denote by $\delta^0(D)$ the minimum over all in- and out-degrees of vertices of D. This is also called the minimum **semi-degree** of a vertex in D. The **arc-connectivity** of D, denoted by $\lambda(D)$, is the minimum out-degree of a proper subset of V. A digraph is **strongly connected** (or just **strong**) if $\lambda(D) \geq 1$. An arc a of a strong digraph D is a **cut-arc** if $D \setminus \{a\}$ is not strong.

When X is a subset of the vertices of a digraph D, we denote by $D[X]$ the subdigraph **induced** by X, that is, the vertex set of $D[X]$ is X and the arc set consists of those arcs of D which have both end vertices in X.

The **independence number**, denoted $\alpha(D)$, of a digraph $D = (V, A)$ is the size of a largest subset $X \subseteq V$ such that the subdigraph of D induced by X has no arcs.

A **strong component** of a digraph D is a maximal (with respect to inclusion) subdigraph D' which is strongly connected. The strong components of D are vertex disjoint and their vertex sets form a partition of $V(D)$. If D has more than one strong component, then we can order these as $D_1, \ldots, D_k$ such that there is no arc from a vertex in $V(D_j)$ to a vertex in $V(D_i)$ when $j > i$. A strong component D_i is **initial** (**terminal**) if there is no arc of D which enters (leaves) $V(D_i)$.

The following result is well-known and easy to show.

Proposition 9. *A digraph D has an out-branching if and only it it has precisely one initial strong component. In that case every vertex of the initial strong component can be the root of an out-branching in D.*

A digraph is **semicomplete** if it has no pair of nonadjacent vertices. A **tournament** is a semicomplete digraph with no directed cycle of length 2. A digraph $D = (V, A)$ is **co-bipartite** is it has a vertex-partition V_1, V_2 such that $D[V_i]$ is semicomplete for $i \in [2]$.

We shall make use of the following classical result due Camion. He formulated it only for tournaments but it is easy to see that it holds for semicomplete digraphs also.

Theorem 10. *[9] Every strongly connected semicomplete digraph is hamiltonian.*

2.1 Non-separating strong spanning subdigraphs

The following conjecture, which would clearly imply Conjecture 3, has been verified for semicomplete digraphs (see Theorem 15 below).

Conjecture 11. [7] There exists a natural number K such that every digraph D with $\lambda(D) \geq K$ contains arc-disjoint spanning strong subdigraphs D_1, D_2.

The infinite family of digraphs described below shows that no condition on semi-degree is enough to imply the conclusion of Conjecture 11 for 2-arc-strong digraphs, even for digraphs with independence number 2.

Proposition 12. *For every natural number K there are infinitely many 2-arc-strong digraphs D with $\alpha(D) = 2$ and $\delta^0(D) \geq K$ that have no pair of arc-disjoint spanning strong subdigraphs.*

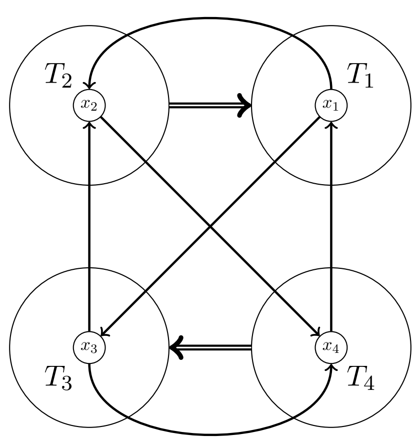

Figure 1: A 2-arc-strong digraph D with $\alpha(D) = 2$ and no decomposition into 2 arc-disjoint spanning subdigraphs.

Proof. Let T be a 2-arc-strong tournament with $\delta^0(T) \geq K$ and let x be a vertex of T. Let $D = (V, A)$ be the digraph that we obtain from 4 disjoint copies T_i, $i \in [4]$, of T by adding the arcs of the 4-cycle $x_1x_3x_2x_4x_1$, the arcs x_1x_2, x_3x_4, all possible arcs from $V(T_2)$ to $V(T_1)$ and from $V(T_4)$ to $V(T_3)$. Here x_i is the copy of x in T_i. See Figure 1. Then D is co-bipartite and 2-arc-strong and we claim that D' does not contain a pair of arc-disjoint spanning strong subdigraphs.

Indeed, suppose there is a partition $A = A_1 \cup A_2$ such that $D_i = (V, A_i)$ is strong for $i = 1, 2$. There are exactly two arcs in D in both directions between $V(T_1) \cup V(T_2)$ and $V(T_3) \cup V(T_4)$. Without loss of generality we have $x_4x_1 \in A_1$ and $x_3x_2 \in A_2$. As there are only two arcs entering $V(T_2)$, this implies that the arc x_1x_2 must be in A_1 (in order to reach the vertices in $V(T_2)$) and as there are only two arcs leaving $V(T_1)$ we have $x_1x_3 \in A_2$. We must also have $x_3x_4 \in A_1$, since the only other arc leaving $V(T_3)$ is in A_2. This implies that the arc x_2x_4 must be in A_2 now we see that there is no path from $V(T_1) \cup V(T_2)$ to $V(T_3) \cup V(T_4)$ in D_1, contradiction. □

2.2 Structure of semicomplete digraphs

Let D be a digraph. A **decomposition** of D is a partition $(S_1, \ldots, S_p)$, $p \geq 1$, of its vertex set. The **index** of vertex v in the decomposition, denoted by $\text{ind}(v)$, is the integer i such that $v \in S_i$. An arc uv is **forward** if $\text{ind}(u) < \text{ind}(v)$, **backward** if $\text{ind}(u) > \text{ind}(v)$, and **flat** if $\text{ind}(u) = \text{ind}(v)$.

A decomposition $(S_1, \ldots, S_p)$ is **strong** if $D\langle S_i \rangle$ is strong for all $1 \leq i \leq p$. The following proposition is well-known (just consider an acyclic ordering of the strong components of D).

Proposition 13. *Every digraph has a strong decomposition with no backward arcs.*

A **nice decomposition** of a strong digraph D is a strong decomposition such that the set of cut-arcs of D is exactly the set of backward arcs. Note that if D has no cut-arc, that is, $\lambda(D) \geq 2$, then the strong decomposition with just one set $S_1 = V(D)$ is nice.

Proposition 14. *[5] Every strong semicomplete digraph of order at least* 4 *admits a nice decomposition.*

Given a semicomplete digraph and a nice decomposition of it, the **natural ordering** of its backward arcs is the ordering of these arcs in decreasing order according to the index of their tail. Note that this ordering is unique [5].

Denote by S_4 the semicomplete digraph on 4 vertices that we obtain from a directed 4-cycle $v_0v_1v_2v_3v_0$ by adding the arcs $v_0v_2, v_2v_0, v_1v_3, v_3v_1$. The following result shows that Conjecture 11 holds for semicomplete digraphs.

Theorem 15. *[7] Let $D = (V, A)$ be a 2-arc-strong semicomplete digraph which is not isomorphic to S_4. Then D contains two arc disjoint strong spanning subdigraphs D_1, D_2.*

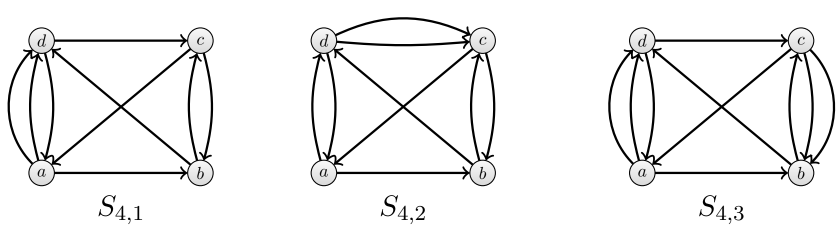

Figure 2: The digraphs $S_{4,1}$, $S_{4,2}$, $S_{4,3}$

Recently Theorem 15 was extended to strong decompositions of 2-arc-strong semicomplete directed multigraphs (parallel arcs allowed).

Theorem 16. *[4] Let D be a 2-arc-strong semicomplete directed multigraph. Then D has a pair of arc-disjoint strong spanning subdigraphs if and only if D is not isomorphic to S_4 or one of three directed multigraphs shown in Figure 2 that can be obtained from S_4 by adding one or two extra arcs parallel to existing ones. Furthermore, if D is not one of those four digraphs, then we can find a pair of arc-disjoint strong spanning subdigraphs in polynomial time.*

2.3 Strong spanning subdigraphs with few arcs in digraphs with $\alpha = 2$.

Theorem 17 (Chen-Manalastras)**.** *[10] Let D be a strongly connected digraph with $\alpha(D) = 2$. Then either D has a hamiltonian cycle or it has two cycles C_1, C_2 that cover $V(D)$ and whose intersection is a (possibly empty) subpath of both cycles.*

Corollary 18. *Let $D = (V, A)$ be a strong digraph with $\alpha(D) = 2$. Then (A) or (B) below holds.*

(A): *$V(D)$ can be partitioned into V_1 and V_2, such that $D[V_i]$ are strong semicomplete digraphs for $i \in [2]$ and there exists $u_i \in V_i$ that is not adjacent to any vertex in V_{3-i}.*

(B): *D has a strong spanning subdigraph S with one of the following properties.*

- **(B1)** *S is a hamiltonian cycle of D.*
- **(B2)** *There are two vertices x, y of S such that $d^+_S(x) = d^-_S(y) = 1$, $d^-_S(x) = d^+_S(y) = 2$ and $d^+_S(z) = d^-_S(z) = 1$ for all $z \in V - \{x, y\}$.*
  Furthermore $N^-_S(x)$ and $N^+_S(y)$ are independent sets in D.
- **(B3)** *There exists a vertex $x \in V$ such that $d^+_S(x) = d^-_S(x) = 2$ and $d^+_S(v) = d^-_S(v) = 1$ for all $v \neq x$.*
  Furthermore $N^+_S(x)$ and $N^-_S(x)$ are independent sets in D.

In particular when one of (B1)-(B3) holds, the sum of the degrees of any two distinct vertices of S is at most 6.

Proof. Let D have $\alpha(D) = 2$. By Theorem 17 D has a hamiltonian cycle or it has two cycles C_1, C_2 that cover $V(D)$ and whose intersection is a (possibly empty) subpath of both cycles. If D has a hamiltonian cycle we take S to be that cycle and we are done as (B1) holds. So now assume that D contains no hamiltonian cycle, which by Theorem 17 implies that D contains two cycles C_1, C_2 that cover $V(D)$ and whose intersection is a (possibly empty) subpath of both cycles. Let such C_1 and C_2 be chosen such that $|V(C_1) \cap V(C_2)|$ is maximum possible.

We now consider the case when $|V(C_1) \cap V(C_2)| > 0$. If $|V(C_1) \cap V(C_2)| = 1$, then let x be the vertex in $V(C_1) \cap V(C_2)$ and let $A(S)$ to be the union of $A(C_1)$ and $A(C_2)$. Now the first part of (B3) holds. If $N^+_S(x)$ is not an independent set, then without loss of generality assume that $xu \in A(C_1)$ and $xv \in A(C_2)$ and $uv \in A(D)$. Now remove the arc xv from C_2 and add the path xuv, in order to obtain a new cycle C_2', with $|V(C_1) \cap V(C_2')| = 2 > 1 = |V(C_1) \cap V(C_2)|$, and thereby contradicting the maximality of $|V(C_1) \cap V(C_2)|$. Therefore $N^+_S(x)$ is an independent set. We can analogously show that $N^-_S(x)$ is an independent set, and therefore part (B3) holds. This completes the case when $|V(C_1) \cap V(C_2)| = 1$. We may therefore assume that $|V(C_1) \cap V(C_2)| \geq 2$. That is, there are vertices $x, y \in V(C_1) \cap V(C_2)$

such that the path common to C_1 and C_2 is P and $P = C_i[x,y]$ for $i = 1,2$. Now the first part of (B2) holds. If $N_S^+(y)$ is not an independent set then analogously to above we get a contradiction to the maximality of $|V(C_1) \cap V(C_2)|$ (or to D not being hamiltonian). And, again analogously to above, we can show that $N_S^-(x)$ is also an independent set. Therefore (B2) holds in this case. This completes the case when $|V(C_1) \cap V(C_2)| > 0$.

Now assume that $|V(C_1) \cap V(C_2)| = 0$ and therefore C_1 and C_2 are vertex disjoint. As D is strongly connected there exists a (C_1, C_2)-arc, say $x_1x_2 \in A(D)$. Let x_1^+ be the successor of x_1 on C_1. If there is any (C_2, x_1^+)-arc, $y_2x_1^+$, in D, then considering the cycle $C_1[x_1^+, x_1]C_2[x_2, y_2]x_1^+$ instead of C_1, would contradict the maximality of $|V(C_1) \cap V(C_2)|$. So there is no (C_2, x_1^+)-arc in D. If there is an (x_1^+, C_2)-arc in D, then consider x_1^+ instead of x_1. Continuing this process either gives us a vertex which is not adjacent to any vertex in C_2 or there is no arc from C_2 to x_1, x_1^+, $\left(x_1^+\right)^+$, etc., a contradiction to D being strong. So there must be a vertex $u_1 \in V(C_1)$ which is not adjacent to any vertex in C_2.

Analogously we can show that there must be a vertex $u_2 \in V(C_2)$ which is not adjacent to any vertex in C_1. This implies that $D[V(C_i)]$ is semicomplete, as if two vertices, x_i, y_i, in $D[V(C_i)]$ are non-adjacent then $\{x_i, y_i, u_{3-i}\}$ is an independent set, a contradiction to $\alpha(D) = 2$.

3 Non-separating out-branchings in semicomplete digraphs

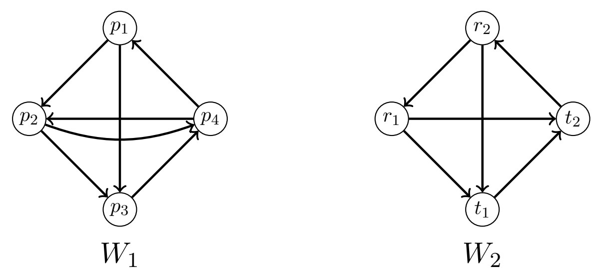

Figure 3: The semicomplete digraphs W_1 and W_2.

Theorem 19. *Let D be a strong semicomplete digraph. Then the following holds.*

(a) If D has at least two vertices with in-degree one, then D contains no non-separating branching. Furthermore if D contains exactly two vertices with in-degree one and is not isomorphic to W_2 (see Figure 3), then there exists an out-tree T^+ rooted at r_1, such that $V(T^+) = V(D) - r_2$ and $D \setminus A(T^+)$ is strong, where $d_D^-(r_1) = d_D^-(r_2) = 1$.

(b) If D is isomorphic to W_1 (see Figure 3), then D contains no non-separating branching.

(c) If D is not isomorphic to W_1 and contains exactly one vertex, r, of in-degree one, then D contains a non-separating branching, rooted at r.

(d) If $\delta^-(D) \geq 2$ and $|V(D)| \leq 3$, then for every $r \in V(D)$ the digraph D contains a non-separating branching, rooted at r.

(e) If $\delta^-(D) \geq 2$ and $|V(D)| \geq 4$, then D admits a nice decomposition $(S_1, S_2, \ldots, S_p)$, and for every $r \in S_1$ the digraph D contains a non-separating branching, rooted at r.

Proof. Recall that in an out-branching every vertex except the root has one arc entering it. Hence if a vertex has in-degree one in D, it must be the root of any non-separating out-branching. This shows that if D admits a non-separating out-branching, it has at most one vertex with in-degree one. This proves the first part of (a).

Now assume that D contains exactly two vertices, r_1 and r_2, with in-degree one and let H be a hamiltonian cycle in D (H exists by Theorem 10) and let $D' = D \setminus A(H)$. If there is only one initial strong component in $D' - r_2$, then letting T^+ be an out-branching in $D' - r_2$ gives us the desired out-tree. So assume that there are at least two initial strong components in $D' - r_2$, $\{r_1\}$ and S_1. If $|V(S_1)| \geq 2$ then as r_1 is non-adjacent to $\{r_2\} \cup V(S_1)$ in D' and $|\{r_2\} \cup V(S_1)| \geq 3$ we obtain a contradiction (as H was a hamiltonian cycle, meaning that we removed only 2 arcs incident to r_1 when we obtained D' from D). So $|S_1| = 1$ and we let $V(S_1) = \{t_1\}$. As $d^-_D(t_1) \geq 2$, we have $d^-_{D'}(t_1) = 1$ and $N^-_{D'}(t_1) = \{r_2\}$. Analogously considering $D' - r_1$ instead of $D' - r_2$ we obtain an initial strong component S_2 in $D' - r_1$ where $V(S_2) = \{t_2\}$ and $N^-_{D'}(t_2) = \{r_1\}$. Note that $t_1 \neq t_2$ (as $r_1t_2, r_2t_1 \in A(D')$). Furthermore t_1 and t_2 are not adjacent in D', as if $t_1t_2 \in A(D')$ then $r_2t_1t_2$ is a path in $D' - r_1$ and S_2 is not an initial strong component in $D' - r_1$. So in D', r_1 is non-adjacent to $\{r_2, t_1\}$ and t_2 is non-adjacent to $\{r_2, t_1\}$. Therefore $|V(D)| = 4$ and D is isomorphic to W_2. This proves the second part of (a).

It is easy to check that if D is the semicomplete digraph W_1 in Figure 3, then every out-branching is separating (the vertex p_1 with in-degree one must be the root of all out-branchings), which proves part (b).

Now suppose that $D = (V, A)$ is different from W_1 and has exactly one vertex of in-degree one, which implies that $n = |V(D)| \geq 3$. Let $H = p_1p_2 \dots p_np_1$ be a hamiltonian cycle in D and let $D' = D \setminus A(H)$. Without loss of generality assume that $d^-_D(p_1) = 1$, which implies that $d^-_{D'}(p_1) = 0$. If D' only has one initial strong component, then D' contains an out-branching, $B^+_{p_1}$, rooted at p_1, which implies that $B^+_{p_1}$ is a non-separating out-branching in D. Therefore we may assume that D' contains at least two initial strong components, one of which is just the vertex p_1. As $d^-_{D'}(x) \geq 1$ for all $x \in V(D') \setminus \{p_1\}$ we note that any other initial strong component, S, in D' must contain at least two vertices. Furthermore as there is no arc between p_1 and the vertices in S in D', we must have $V(S) = \{p_2, p_n\}$ and $p_2p_n, p_np_2 \in A(D')$. Therefore $n \geq 4$, as otherwise $p_2p_3 \in A(H)$ and $p_2p_3 \in A(D')$, a contradiction.

If $n = 4$, then we note that $D = W_1$, a contradiction (as $H = p_1p_2p_3p_4p_1$ and $A(D') = \{p_2p_4, p_4p_2, p_1p_3\}$). So assume that $n \geq 5$. Let T be obtained from H by deleting the arc p_1p_2 and adding the arcs p_np_2 and p_1p_3. Note that T is a strongly connected spanning subgraph of D. Let $D^* = D \setminus A(T)$, and note that $p_1p_i \in A(D^*)$ for all $i \in [n] \setminus \{n, 3\}$ and $p_1p_2, p_2p_n, p_np_3 \in A(D^*)$ (as the vertex set of the initial component, S, in D' was $\{p_2, p_n\}$). Therefore the only initial component in D^* is $\{p_1\}$ and there exists an out-branching, $B^+_{p_1}$ in D^* rooted at p_1, which is therefore a non-separating branching in D. This proves part (c).

We now consider the case when $\delta^-(D) \geq 2$. If $n \leq 3$, then D is the complete digraph on three vertices and part (d) holds. So assume that $n \geq 4$. By Proposition 14, D admits a nice decomposition $(S_1, S_2, \dots, S_p)$.

First consider the case when $p = 1$. That is D is 2-arc-strong. If D is isomorphic to S_4 (see Theorem 15), then as can be seen in Figure 4, S_4 has a non-separating branching B^+_r for each $r \in V(S_4)$. So we may assume that D is not isomorphic to S_4, which by Theorem 15 implies that D contains two arc disjoint strong spanning subdigraphs D_1 and D_2. For every $r \in V(D)$ we note that D_1 contains an out-branching rooted at r and therefore D contains a non-separating branching, rooted at r. This proves part (e) when $p = 1$.

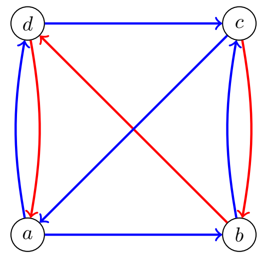

Figure 4: Decomposing S_4 into an out-branching B_c^+ in red and a strong spanning subdigraph in blue.

We now consider the case when $p \geq 2$. Let $r \in S_1$ be arbitrary. Let st be the $(V(D) \setminus V(S_1), V(S_1))$-arc in D. That is, st, is the cut-arc entering S_1.

Construct a new digraph H_r from S_1 by adding a vertex x to S_1 and adding the two arcs xt and xr (if $t = r$ we add two parallel arcs). We will first show that x has two arc-disjoint paths to every other vertex in H_r. As S_1 is strong we note that x can reach all other vertices in H_r if we delete xt or xr. Furthermore if we delete any arc $e \in A(S_1)$ then x can still reach all other vertices in S_1 by starting with the arc xt, as t can reach all other vertices in S_1 even after deleting one arc (by the definition of a nice decomposition). By Theorem 1 this implies that there exists two arc-disjoint out-branchings both rooted in x in H_r. Deleting x from these gives us two arc-disjoint out-branchings B_t^+, B_r^+ in S_1, rooted at t and r, respectively.

We will now show that we may assume that B_t^+ is not just an out-star rooted at t. Assume that B_t^+ is an out-star rooted at t. We first consider the case when $|S_1| \geq 4$. Let $l_1, l_2, \ldots, l_{|S_1|-1}$ be the leaves of B_t^+ and note that $\{l_1, l_2, \ldots, l_{|S_1|-1}\}$ is not independent in $D \setminus A(B_r^+)$ as the underlying graph of B_r^+ is acyclic. So without loss of generality we may assume that $l_1 l_2 \in D \setminus A(B_r^+)$. Now delete the arc tl_2 from B_t^+ and add the arc $l_1 l_2$ instead. We have then obtained a B_t^+ (arc disjoint from B_r^+) that is not an out-star, as desired. So we may now consider the case when $|S_1| \leq 3$. Notice that $|S_1| = 1$ is impossible as $\delta^-(D) \geq 2$, so we have $|S_1| \geq 2$. However if $|S_1| = 2$, denoting S_2 by $\{t, y\}$, we have $d_D^-(y) = 1$, a contradiction. So we must have $|S_1| = 3$. Let $S_1 = \{t, x, y\}$. As $d_D^-(y), d_D^-(x) \geq 2$ we note that $xy, yx, tx, ty \in A(D)$. As S_1 is strong we note that $xt \in A(D)$ or $yt \in A(D)$ (or both). Without loss of generality assume that $xt \in A(D)$. We now obtain the desired B_t^+ and B_r^+ as follows.

- If $r = t$, then $B_t^+ = \{tx, xy\}$ and $B_r^+ = \{ty, yx\}$.
- If $r = x$, then $B_t^+ = \{ty, yx\}$ and $B_r^+ = \{xt, xy\}$.
- If $r = y$, then $B_t^+ = \{tx, xy\}$ and $B_r^+ = \{yx, xt\}$.

Now, as B_t^+ is not just an out-star it contains a vertex q which is neither the root or leaf. As D is a strong semicomplete digraph it contains a hamiltonian cycle, $H = p_1 p_2 p_3 \ldots p_n p_1$. Without loss of generality assume that the cut-arc st is the arc $p_n p_1$. Then there must be exactly one arc in H leaving S_1, say $p_i p_{i+1}$. Note that $p_1 p_2 \ldots p_i$ is a hamiltonian path in S_1 and $p_{i+1} p_{i+2} \ldots p_n$ is a hamiltonian path in $D - V(S_1)$. Let Q be the union of B_t^+ and the path $p_{i+1} p_{i+2} \ldots p_n p_1$ where we add an arc from every leaf of B_t^+ to p_{i+1} (which exists by the definition of the nice decomposition and the fact that t is not a leaf of B_t^+). Note that Q is a strong spanning subdigraph of D.

Now construct B^+ by starting with B_r^+ and adding an arc from q (the vertex that was not the root or a leaf of B_t^+) to every vertex in $V(D) \setminus V(S_1)$. Note that B^+ is an out-branching in D rooted at r and $D - A(B^+)$ contains all arcs of Q and is therefore strongly connected. This completes the proof of part (e) and therefore also of the theorem. □

As the digraph S_4 has a non-separating out-branching B_v^+ for each of its 4 vertices, the same holds for any digraph obtained from S_4 by adding arcs parallel to existing ones. Thus we one can prove the following corollary of Theorem 16.

Corollary 20. *Every 2-arc-strong semicomplete directed multigraph $D = (V, A)$ has a non-separating out-branching B_v^+ for every choice of $v \in V$.*

□

4 Proof of Theorem 8

Before we prove Theorem 8 we need the following lemma.

Lemma 21. *Let D have $\alpha(D) = 2 \leq \lambda(D)$ and assume that $\delta^-(D) \geq 3$. If D satisfies (A) in Corollary 18, then D has a non-separating out-branching.*

Proof. Let D be a digraph with $\alpha(D) = 2 \leq \lambda(D)$ which consists of vertex disjoint strong semicomplete digraphs D_1, D_2, such that there exists $u_i \in V(D_i)$ that is not adjacent to any vertex in D_{3-i} for $i = 1, 2$. As $\delta^-(D) \geq 3$ we note that $d^-_{D_i}(u_i) \geq 3$. Therefore $|V(D_1)|, |V(D_2)| \geq 4$ and neither D_1 nor D_2 is isomorphic with W_1 or W_2 (see Figure 3). We will now construct in D an out-branching, B^+, and a spanning strong subdigraph, Q, which are arc-disjoint, as follows. To start this construction, consider the following three cases for $i = 1, 2$.

Case 1. There are at least two vertices of in-degree one in D_i.
As $|V(D_i)| \geq 4$ and D_i is a strong semicomplete digraph, we note that there are exactly two vertices, r_1^i and r_2^i, of in-degree one (there can be at most 3 vertices of in-degree one in a semicomplete digraph and if there were 3 such vertices in D_i, then it would not be strong). By Theorem 19 and the fact that D_i is not isomorphic to W_2, there exists an out-tree $T^+_{r_i}$ rooted at r_1^i and spanning $V(D_i) \setminus \{r_2^i\}$ such that $D_i \setminus A(T^+_{r_i})$ is strongly connected. Add the arcs of $T^+_{r_i}$ to B^+ and add the arcs of $D_i \setminus A(T^+_{r_i})$ to Q. As $\delta^-(D) \geq 3$ we note that there exists at least two (D_{3-i}, r_1^i)-arcs and at least two (D_{3-i}, r_2^i)-arcs in D.

Case 2. There is exactly one vertex, r^i, of in-degree one in D_i.
As D_i is not isomorphic to W_1, Theorem 19 implies that there is a non-separating out-branching, $B^+_{r^i}$, in D_i, rooted at r^i. In this case add the arcs of $B^+_{r^i}$ to B^+ and the remaining arcs of D_i to Q. As $\delta^-(D) \geq 3$ we note that there exists at least two (D_{3-i}, r^i)-arcs in D.

Case 3. $\delta^-(D_i) \geq 2$.
As $|V(D_i)| \geq 4$, then Theorem 19 (e) implies that D_i admits a nice decomposition $(S_1^i, S_2^i, \ldots, S_{p_i}^i)$, and for every $r' \in S_1^i$ the digraph D_i contains a non-separating branching, rooted at r'. As $\lambda(D) \geq 2$, there must be a (D_{3-i}, S_1^i)-arc, ur^i, in D. Let $B^+_{r^i}$ be a non-separating branching, rooted at r^i in D_i. Add the arcs of $B^+_{r^i}$ to B^+ and the remaining arcs of D_i to Q.

This completes our three cases. Note that Q contains a strong spanning subdigraph of D_1 and of D_2. Furthermore in cases 2 and 3, B^+ contains an out-branching of D_i rooted at a vertex r^i, such that there exists a (D_{3-i}, r^i)-arc in D. In Case 1, B^+ consists of an out-tree, rooted at r_1^i, containing all vertices of D_i except r_2^i, such that both r_1^i and r_2^i have at least two arcs into them from D_{3-i}. We now consider the following possibilities.

We were in Case 2 or 3 for both D_1 and D_2. Add an arc from D_1 to the root of the out-branching of D_2 to B^+. As $\lambda(D) \geq 2$, we can add a further (D_1, D_2)-arc and a (D_2, D_1)-arc to Q, in order for B^+ and Q to fulfill the desired properties.

We were in Case 2 or 3 for D_1 and Case 1 for D_2. Add an arc from D_1 to r_1^2 and to r_2^2. As there were at least two (D_1, r_2^2)-arcs in D we can add a further (D_1, r_2^2)-arc to Q and as $\lambda(D) \geq 2$, we can add a (D_2, D_1)-arc to Q. Now B^+ and Q fulfill the desired properties.

We were in Case 2 or 3 for D_2 and Case 1 for D_1. This case is analogous to the previous case.

We were in Case 1 for both D_1 and D_2. Add an arc from $V(D_1) \setminus \{r_2^1\}$ to r_1^2 and to r_2^2 to B^+ (which is possible as r_1^2 and r_2^2 have at least two arcs into them from D_1). Also add an arc from $V(D_2) \setminus \{r_2^2\}$ to r_2^1 to B^+. Now B^+ is an out-branching rooted at r_1^1 in D. As there are at least two arcs into r_1^i and into r_2^i from D_{3-i} we note that there are at least four (D_2, D_1)-arcs and at least four (D_1, D_2)-arcs in D. We can therefore add a (D_2, D_1)-arc and a (D_1, D_2)-arc to Q such that Q and B^+ are arc-disjoint. Now B^+ and Q fulfill the desired properties, completing the proof of the theorem. □

Let us recall Theorem 8.

Theorem 8. *Let D be a 2-arc-strong digraph with $\alpha(D) \leq 2$. If either of the following statements hold then there exists an out-branching, B^+, in D, such that $D \setminus A(B^+)$ is strongly connected.*

(i): $\delta^-(D) \geq 5$, *or*

(ii): $\delta^-(D) \geq 3$ *and D is oriented (has no 2-cycle).*

Proof. Let D be a 2-arc-strong digraph with $\alpha(D) \leq 2$. By Theorem 19 we may assume that $\alpha(D) = 2$. By Lemma 21, we may assume that D has a strong spanning subdigraph H satisfying one of the conditions (B1)-(B3) in Case B of Corollary 18. Let $D' = D \setminus A(H)$.

If D' has only one initial strong component, then, by Proposition 9 there is an out-branching in D' and the theorem is proved. So we may assume that $R'_1, R'_2, \ldots, R'_t$ are the initial strong components in D' and $t \geq 2$. For all $i \in [t]$, let $R_i = D\langle V(R'_i)\rangle$. We will now prove the following claims.

Claim A: *$t = 2$ and $|V(R_1)|, |V(R_2)| \geq 5$. Furthermore, all in-degrees in D' are at least two, except possibly for one vertex whose indegree is at least one. That is, there exists $r \in V(D')$, such that $d^-_{D'}(r) \geq 1$ and $d^-_{D'}(x) \geq 2$ for all $x \in V(D') \setminus \{r\}$.*

In Case (i) we actually have $\delta^-(D') \geq 3$.

Proof of Claim A: First consider Case (i) (when $\delta^-(D) \geq 5$). As $\Delta^-(H) \leq 2$ we note that $\delta^-(D') \geq 3$, and therefore also $\delta^-_{D'}(R_1) \geq 3$. This further implies that $|V(R_i)| = |V(R'_i)| \geq 4$, since $N^-_{D'}[x] \subseteq V(R'_i)$, for all $x \in V(R'_i)$ and $i \in [t]$. Now noting that there is at most one vertex of H whose in-degree is more than 1, we see that R'_i contains a vertex with at least 4 in-neighbours inside R'_i so $|V(R_i)| = |V(R'_i)| \geq 5$ holds.

Now consider the Case (ii). In this case we note that all in-degrees in D' are at least two, except possibly for one vertex whose indegree is at least one. Let $n_i = |V(R_i)|$ and note that the number of arcs in R_i is at least $2n_i - 1$ (as all arcs into a vertex in R_i belong to R_i). As D is oriented this implies that $\binom{n_i}{2} \geq |E(V_i)| \geq 2n_i - 1$. As $\binom{n_i}{2} < 2n_i - 1$ if $n_i \in [4]$, we must have $n_i \geq 5$, which implies that $|V(R_i)| \geq 5$ in all cases.

For the sake of contradiction assume that $t \geq 3$. Let $x_1 \in V(R_1)$ be an arbitrary vertex with $d_H^+(x_1) = d_H^-(x_1) = 1$ and let N_1 be the set of the two neighbours of x_1 in H. As $|V(R_2)| \geq 5$ there are at least 3 vertices in $V(R_2)$ that are not in N_1. By part (B) in Corollary 18 we note that at most two vertices in H have degree more than two, so we can can choose a vertex $x_2 \in V(R_2) \setminus N_1$ such that $d_H^+(x_2) = d_H^-(x_2) = 1$. Let N_2 be the set of the two neighbours of x_2 in H. Now there exists a vertex $x_3 \in V(R_3) \setminus (N_1 \cup N_2)$ which implies that $\{x_1, x_2, x_3\}$ is an independent set in D, contradicting $\alpha(D) \leq 2$. Therefore $t = 2$, which completes the proof of Claim A.

Claim B: *R_1 and R_2 are semicomplete digraphs. Furthermore, for all $z \in V(D)$ the following holds,*

$$|N_H^+(z) \cap V(R_1)|, |N_H^+(z) \cap V(R_2)|, |N_H^-(z) \cap V(R_1)|, |N_H^-(z) \cap V(R_2)| \leq 1$$

Proof of Claim B: For the sake of contradiction assume that $x_1, y_1 \in V(R_1)$ and x_1 and y_1 are non-adjacent in D. Let $N_2 = (N_H^+(x_1) \cup N_H^-(x_1) \cup N_H^+(y_1) \cup N_H^-(y_1)) \cap V(R_2)$. If $V(R_2) \not\subseteq N_2$, then let $z_2 \in V(R_2) \setminus N_2$ and note that $\{x_1, y_1, z_2\}$ is independent in D, contradicting that $\alpha(D) \leq 2$. So, $V(R_2) \subseteq N_2$, which implies that $|N_2| \geq |V(R_2)| \geq 5$, by Claim A. By Corollary 18, we note that $d_H(x_1) + d_H(y_1) \leq 6$, which implies the following,

$$6 \geq |N_H^+(x_1) \cap V(R_2)| + |N_H^-(x_1) \cap V(R_2)| + |N_H^+(y_1) \cap V(R_2)| + |N_H^-(y_1) \cap V(R_2)| \geq |N_2| \geq 5$$

First consider the case when $|N_H^+(x_1) \cap V(R_2)| \geq 2$. By the construction of H we note that $|N_H^+(x_1) \cap V(R_2)| = 2$, so let $N_H^+(x_1) \cap V(R_2) = \{x_2, y_2\}$. By Corollary 18, x_2 and y_2 are non-adjacent in D. As $\alpha(D) = 2$ we note that either x_2 or y_2 has to be adjacent to y_1. Without loss of generality assume that y_2 is adjacent to y_1. As y_2 is adjacent to both x_1 and y_1 in D we note that we must have $d_H(x_1) + d_H(y_1) = 6$, as H satisfies one of (B2), (B3) in Corollary 18 (and $|N_2| = |V(R_2)| = 5$).

If $|N_H^-(x_1) \cap V(R_2)| \geq 2$ or $|N_H^+(y_1) \cap V(R_2)| \geq 2$ or $|N_H^-(y_1) \cap V(R_2)| \geq 2$ then we analogously can show that $d_H(x_1) + d_H(y_1) = 6$ and there exists two non-adjacent vertices x_2 and y_2 in R_2.

If we had considered $\{x_2, y_2\}$ instead of $\{x_1, y_1\}$ then we would analogously have obtain $d_H(x_2) + d_H(y_2) = 6$. However, by part (B) in Corollary 18 we note that it is not possible to have vertex-disjoint sets, $\{x_1, y_1\}$ and $\{x_2, y_2\}$, such that $d_H(x_1) + d_H(y_1) = 6 = d_H(x_2) + d_H(y_2)$. Therefore R_1 is semicomplete. Analogously we can show that R_2 is also a semicomplete digraphs.

Let $z \in V(D)$ be arbitrary. For the sake of contradiction assume that $|N_H^+(z) \cap V(R_1)| \geq 2$. By Corollary 18 we must have $|N_H^+(z) \cap V(R_1)| = 2$, so let $N_H^+(z) \cap V(R_1) = \{x_1, y_1\}$. By Corollary 18 x_1 and y_1 are non-adjacent in D. This contradicts the fact that R_1 is semicomplete. Therefore $|N_H^+(z) \cap V(R_1)| \leq 1$. Analogously $|N_H^+(z) \cap R_2|, |N_H^-(z) \cap R_1|, |N_H^-(z) \cap R_2| \leq 1$, which completes the proof of Claim B.

Claim C: *Let $y \in V(D) \setminus (V(R_1) \cup V(R_2))$ be arbitrary. If y has at most one arc entering it from $V(R_1)$ in D, then y is adjacent to all vertices in $V(R_2)$ and furthermore has at least four arcs entering it from $V(R_2)$.*

Analogously, if y has at most one arc entering it from $V(R_2)$ in D, then y is adjacent to all vertices in $V(R_1)$ and furthermore has at least four arcs entering it from $V(R_1)$.

The above implies that every vertex in $V(D) \setminus (V(R_1) \cup V(R_2))$ has at least four arcs entering it from $V(R_1) \cup V(R_2)$ in D.

Proof of Claim C: Assume that $y \in V(D) \setminus (V(R_1) \cup V(R_2))$ and y has at most one arc entering it from $V(R_1)$ in D. For the sake of contradiction assume that there exists $r_2 \in V(R_2)$ which is non-adjacent to y (in D). By Claim B, we note that $|N_H^+(r_2) \cap V(R_1)|, |N_H^-(r_2) \cap V(R_1)|, |N_H^+(y) \cap V(R_1)|, |N_H^-(y) \cap V(R_1)| \leq 1$. As R_1' and R_2' are initial strong components in D', and therefore all arcs between r_2 and R_1 in D belong to H, we note that r_2 is adjacent to at most two vertices in R_1 (in D). As y has at most one arc entering it from $V(R_1)$ in D and at most one arc from y to R_1 in H (and therefore also in D, as R_1' is an initial strong components in D'), we note that y is adjacent to at most two vertices in R_1 (in D). As $|V(R_1)| \geq 5$, by Claim A, this implies that there exists a $r_1 \in V(R_1)$ which is not adjacent to r_2 or y in D, contradicting $\alpha(D) = 2$. Therefore y is adjacent to every vertex in R_2.

As R_2' was an initial strong component in D' and $y \notin V(R_2')$ we note that every arc from y to R_2 in D belongs to H. By Claim A and Claim B, we note that $|N_H^+(y) \cap V(R_2)| \leq 1$ and $|V(R_2)| \geq 5$, which implies that there are at least four arcs from $V(R_2)$ to y in D. This completes the first part of the proof of Claim C. The second part is proven analogously (by swapping the names of R_1 and R_2).

Let $x \in V(D) \setminus (V(R_1) \cup V(R_2))$ be arbitrary. If x has less than two arcs entering it from R_i then it has four arcs entering it from R_{3-i} ($i \in [2]$). And if x has at least two arcs entering it from both R_1 and from R_2, then it also has at least four arcs entering it from $V(R_1) \cup V(R_2)$. This completes the proof of Claim C.

Construction of R_1^*. Initially let $R_1^* = R_1$. Now for every $u \in V(D) \setminus (V(R_1^*) \cup V(R_2))$ with at least one arc into R_1^* in D and at most one arc from R_2 to u, add u to R_1^*. Continue this process until no further vertex can be added.

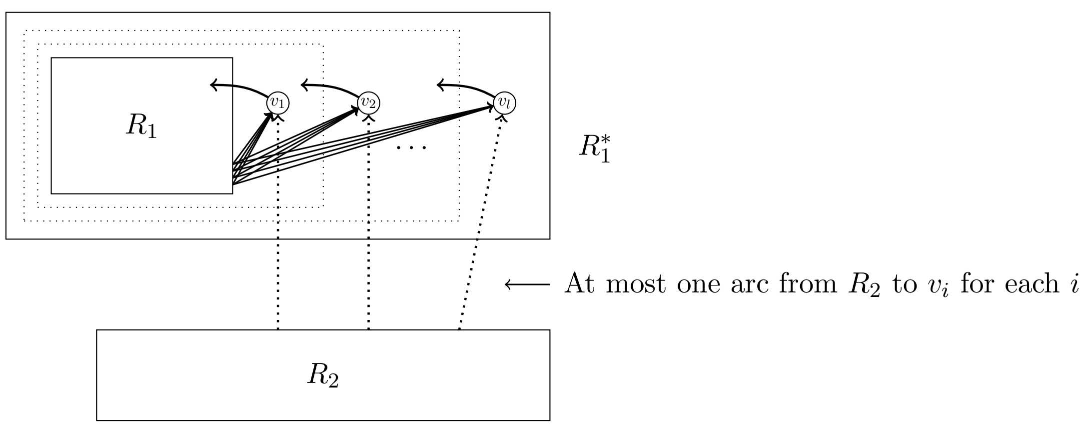

Figure 5: An illustration of the construction of R_1^*, where R_1^* is constructed from R_1 by adding the vertices $v_1, v_2, \ldots, v_l$ in that order. Every v_i has at least one arc into $R_1 \cup \{v_1, v_2, \ldots, v_{i-1}\}$. By Claim C there are at least four arcs from R_1 into v_i for each $i \in [l]$.

Claim D: *R_1^* is a strong semicomplete digraph.*

Proof of Claim D: Let $Q = V(R_1^*) \setminus V(R_1)$. That is, Q denotes the set of vertices added to R_1 in the construction of R_1^*. By construction, when it was added to the current R_1^* each such vertex had at least one arc into the current set $V(R_1^*)$ and at most one arc entering it from $V(R_2)$ in D. We will first show that R_1^* is semicomplete. Assume for the sake of contradiction that $q_1, q_2 \in V(R_1^*)$ are non-adjacent in D. By Claim B we note that q_1 and q_2 cannot both belong to $V(R_1)$. By part 2 of Claim C we note that we cannot have $q_1 \in V(R_1)$ and $q_2 \in Q$ (or vice versa), which implies that we must have $q_1, q_2 \in Q$. This implies that q_1 and q_2 both have at most one arc into them from $V(R_2)$. By Claim B, $|N_H^+(q_1) \cap V(R_2)|, |N_H^+(q_2) \cap V(R_2)| \leq 1$, which implies that $|N_D^+(q_1) \cap V(R_2)|, |N_D^+(q_2) \cap V(R_2)| \leq 1$. Therefore each of q_1 and q_2 are adjacent to at most two vertices in R_2. As $|V(R_2)| \geq 5$, there is a vertex $r_2 \in V(R_2)$ which is non-adjacent to both q_1 and q_2 in D, contradicting that $\alpha(D) \leq 2$. Therefore R_1^* is semicomplete.

As R_1 is strongly connected and every vertex we add in the process of building R_1^* has an arc into the current set R_1^* and an arc (actually at least 4 arcs) entering it from R_1 (and $R_1 \subseteq R_1^*$), by Claim C, we note that the current set R_1^* is strongly connected in every step of the construction. Therefore the final R_1^* is also strongly connected. This completes the proof of Claim D.

Definitions. By Claim A and D and Proposition 14 we note that R_1^* has a nice decomposition $(S_1, \ldots, S_p)$. If $p \geq 2$ then there is only one arc entering S_1 in R_1^* so let uv be an arc entering S_1 in D, which does not belong to R_1^*. Such an arc exists as D is 2-arc-strong. If $p = 1$ then R_1^* is 2-arc-strong. In this case let uv be any arc entering R_1^* in D. Let $D^* = D' - uv$ (that is, delete the arc uv from D').

Claim E: *There exists an out-branching B_1^+ in R_1^* rooted at v, such that $R_1^* - A(B_1^+)$ is strongly connected.*

Furthermore there exists an out-branching B_2^+ in R_2, such that $R_2 - A(B_2^+)$ is strongly connected.

Proof of Claim E: As R_1^* is strongly connected by Claim D, We can apply Theorem 19 to R_1^*. By Claim A we note that $|V(R_1^*)| \geq 5$ and therefore R_1^* is not isomorphic to W_1.

Also, by Claim A we note that all vertices of R_1, except possibly one, have in-degree at least two in R_1. As every vertex we add to R_1 in the construction of R_1^* have in-degree at least four (from R_1, by Claim C), we note that all vertices of R_1^*, except possibly one, have in-degree at least two in R_1^*. Therefore we are in case (c)-(e) of Theorem 19.

Note that if there is a vertex of in-degree one in R_1^*, then v is that vertex. Furthermore $v \in S_1$ (where S_1 was defined just above Claim E, as part of the nice decomposition on R_1^*). Therefore by Theorem 19 we note that R_1^* contains a non-separating out-branching rooted at v, which completes the first part of the proof of Claim E.

The second part of Claim E, follows analogously, by Theorem 19.

Completion of the proof. Let $B^+ = B_1^+ \cup B_2^+$ (defined in Claim E) and let $Q = (R_1^* - A(B_1^+)) \cup (R_2 - A(B_2^+))$ and let $V^* = V(R_1^*) \cup V(R_2)$. Note that B^+ consists of two vertex-disjoint out-trees (whose union span V^*) and Q of two vertex-disjoint strong components (whose union also span V^*). Let P_{12} be a path from $V(R_1^*)$ to $V(R_2)$ in D^* and let P_{21} be a path from $V(R_2)$ to $V(R_1^*)$ in D^* Add the arcs of P_{12} and P_{21} to Q. For every vertex $p \in (V(P_{12}) \cup V(P_{21})) \setminus V^*$ do the following. Add an arc, which is not in $A(P_{12}) \cup A(P_{21})$, from $V(R_1^*) \cup V(R_2)$ to p to B^+. This is possible by Claim C and the fact that there are at most two arcs in $A(P_{12}) \cup A(P_{21})$ leaving $V(R_1^*) \cup V(R_2)$ (at most one leaves $V(R_1^*)$ and at most one leaves $V(R_2)$). Furthermore if $p = u$ (recall that the arc uv was defined above Claim E), then make sure the added arc leaves $V(R_2)$ (and not $V(R_1^*)$), which is possible as u was not added to R_1^* and therefore has at least two arcs into it from $V(R_2)$ (here we used that the arc uv enters R_1^*). Finally add p to V^*.

When this process is completed $V^* = V(R_1^*) \cup V(R_2) \cup V(P_{12}) \cup V(P_{21})$, Q induces a strong subgraph on the vertex set V^* and B^+ still consist of two out-trees also spanning V^*, one of which is rooted at v. Furthermore the arc uv is not used above and if $u \in V^*$ then it belongs to the out-tree not rooted at v. We now add the remaining vertices as follows. While $V^* \neq V(D)$ let P' be any (V^*, V^*)-path in D^* with at least one internal vertex. We can construct P' by letting p_0p_1 be any arc out of V^* in D^* and then taking any path from p_1 back to V^* in D^*. Now add $A(P')$ to Q and for every vertex $p \in V(P') \setminus V^*$ do the following (analogously to above). Add an arc, which is not in $A(P')$, from $V(R_1) \cup V(R_2)$ to p to B^+, which is possible by Claim C and the fact that there is at most one arc in $A(P')$ leaving $V(R_1) \cup V(R_2)$. Furthermore if $p = u$ (recall that the arc uv was defined above Claim E), then make sure the added arc leaves $V(R_2)$ (and not $V(R_1^*)$), which is possible as u was not added to R_1^* and therefore has at least two arcs into it from $V(R_2)$ (here we used that the arc uv enters R_1^*). Finally add p to V^*.

We continue the above process until $V^* = V(D)$. Now Q is a strong spanning subgraph of D, which does not include any arcs from B^+ and also does not include the arc uv. B^+ consists of two out-trees, one rooted at v and u belonging to the out-tree which was not rooted at v. Therefore by adding the arc uv to B^+ we obtain a spanning out-branching of D which is arc-disjoint to Q, thereby completing the proof. □

5 Non separating spanning trees in digraphs with independence number 2

Let us recall that for a subdigraph H of D, we denote by $D - H$ the sudigraph of D obtained from removing the vertices of H from D, that is $D - H = D[V(D) \setminus V(H)]$. Furthermore we denote by $D \setminus A(H)$ the sudigraph of D obtained from removing the arcs of H from D, that is $V(D - H) = V(D)$ and $A(D - H) = A(D) \setminus A(H)$.

A spanning tree T of a connected digraph D is **safe** if for every pair of distinct vertices x and y of D, there exists an oriented path from x to y in D if and only if there exists also an oriented path from x to y in $D \setminus A(T)$. In particular, a safe spanning tree of a strong digraph is a non separating spanning tree.

At several places, we use the following fact. Assume that H is an induced subdigraph of D such that H admits a safe spanning tree T, and assume also that there exist $u, v \in H$ and $x \in D \setminus H$ such that $ux, vx \in A(D)$ and that there exist a path from u to v in H. Then $D[V(H) \cup x]$ admits the safe spanning tree $T + ux$. Indeed, there exists also a path from u to v in $H \setminus A(T)$ and thus a path from u to x in $D[V(H) \cup x] \setminus (A(T) \cup \{ux\})$.

First, we derive some results on safe spanning trees of semicomplete digraphs.

Lemma 22. *Every semicomplete digraph on at least five vertices admits a safe spanning tree.*

Proof. Let D be a semicomplete digraph on at least five vertices. If D is strong, as D contains at least five vertices, then we find a spanning tree in the complement of a Hamiltonian cycle of D. This spanning tree is clearly safe.

So, assume that D is not strong and denote by $C_1, C_2, \dots, C_t$ the strongly connected components of D such that there is no arc from C_i to C_j if $i > j$. We denote by K the subdigraph of D containing the vertices $V(D)$ and the set of arcs of D connecting its strong components (that is the arcs uv with $u \in C_i$, $v \in C_j$ and $i \neq j$). Moreover, for every $i = 1, \dots, t$ let x_i be a vertex of C_i and P be the path $x_1 \dots x_t$. If there exists a spanning tree T of D all of whose arcs are in the subdigraph $K' = K \setminus A(P)$, then T is a safe spanning tree of D. Thus we have to check that K' is a connected subdigraph of D.

First, if there exist i and j such that $|C_i| \geq 2$ and $|C_j| \geq 2$, then we pick a vertex y_i in C_i different from x_i and a vertex y_j in C_j different from x_j. In K', every vertex not in C_i is adjacent to y_i and every vertex in C_i is adjacent to y_j. So, K' is connected in this case.

Moreover, if $t \geq 4$ then in K' every vertex of $D \setminus C_1 \cup C_2$ is adjacent to x_1 and every vertex of $D \setminus C_{t-1} \cup C_t$ is adjacent to x_t. So, K' is connected.

Thus we may assume that $t \leq 3$ and that there exists $i_0 \in \{1, \ldots, t\}$ such that for every $i \neq i_0$, the component C_i has size 1 exactly. If $t = 3$, then, as D contains at least 5 vertices, we have $|V(C_{i_0})| \geq 3$. If $i_0 = 1$, then K' contains all the arcs from $V(C_1) - x_1$ to $\{x_2, x_3\}$ and the arc x_1x_3. So K' is connected. The case $i_0 = 3$ is symmetrical.

So we may assume that $i_0 = 2$. Let $y_2 \in V(C_2) \setminus \{x_2\}$ be arbitrary and let $K^* = K \setminus \{x_1x_2, y_2x_3\}$. As every vertex in $V(C_2)$ is adjacent to x_1 or x_3 in K^* and $x_1x_3 \in A(K^*)$ we note that K^* is a spanning connected subgraph of D and we can therefore in K^* find a safe spanning tree of D.

Finally, if $t = 2$, by symmetry again we can assume that $i_0 = 1$. We have $|V(C_1)| \geq 4$ and so C_1 contains an arc xy such that $C_1 \setminus xy$ is strongly connected. In this case the arcs from $V(C_1) \setminus \{x\}$ to x_2 plus the arc xy form the arcs of a safe spanning tree of D. □

The following claim will be useful in the proof of the main theorem of the section.

Claim 22.1. *Let D be a digraph with $\lambda(D) \geq 2$. If D contains a tree T such that $D \setminus A(T)$ is strongly connected and $V(D) \setminus V(T)$ has size at most two and induces a semicomplete digraph, then D admits a non-separating spanning tree.*

Proof. Let D and T as stated, and let C be a terminal strong component of $D[V(T)] \setminus A(T)$.

Suppose first that $V(D) \setminus V(T) = \{x\}$. As $\lambda(D) \geq 2$, there are at least two arcs from C to x. Let be u an in-neighbour of x in C. The digraph $D \setminus (A(T) \cup \{ux\})$ is still strong and so $T + ux$ is a non-separating spanning tree of D.

Now, assume that $V(D) \setminus V(T) = \{x, y\}$. If there are two arcs ux and vx from C to x, then $T' = T + ux$ is a tree on $n - 1$ vertices such that $D \setminus A(T')$ is strongly connected and we can conclude from the previous case that D admits a separating spanning tree. So, as $\lambda(D) \geq 2$, there are at least two arcs from C to $\{x, y\}$ and we can assume that one, say ux, has head x and the other, say vy, has head y. Similarly, we can assume by the previous case, that if C' is an initial strong component of $D[V(T)] \setminus A(T)$, then there exist two arcs xu' and yv' with $u', v' \in C'$. As $D[\{x, y\}]$ is semicomplete, we can assume without loss of generality that xy is an arc of D. Now it is easy to check that $D \setminus (A(T) \cup \{vy, xu'\})$ is strongly connected and that D admits the non-separating spanning tree $T + vy + xu'$. □

Now we can prove the following.

Theorem 23. *Every digraph $D = (V, A)$ with $\alpha(D) \leq 2 \leq \lambda(D)$ such that D contains a semicomplete digraph on at least 5 vertices has a non separating spanning tree. In particular, every digraph $D = (V, A)$ with $\alpha(D) \leq 2 \leq \lambda(D)$ such that $|V| \geq 14$ has a non-separating spanning tree.*

Proof. If D is semicomplete, then the result follows from Lemma 22. So we may assume that $\alpha(D) = 2$. As the Ramsey number $R(3, 5)$ is 14 [15] and $\alpha(D) = 2$, it follows that if $|V| \geq 14$, then D contains a semicomplete subdigraph on five vertices. Hence we may assume below that D_1 is a semicomplete digraphs of D on 5 vertices. By Lemma 22, D_1 contains a safe spanning tree. So let R be a maximal induced subdigraph of D containing $V(D_1)$ and admitting a safe spanning tree. We now show that $R = D$. Suppose for a contradiction that this is not the case.

Let T be a safe spanning tree of R and consider a vertex x of $S = D[V \setminus V(R)]$. The vertex x has at most one in-neighbour in D_1. Indeed, otherwise, assume that y and z are two in-neighbours of x in D_1 with yz being an arc of D_1 (recall that D_1 is semicomplete). But then, $T + yx$ would be a safe spanning tree of $D[V(R) \cup x]$, a contradiction to the maximality of R. Similarly, x has at most one out-neighbour in D_1. So we can conclude that S is a semicomplete subdigraph of D. Indeed, otherwise, S would contain an independent set $\{u, v\}$ of size two. But as u and v have each at most two neighbours in D_1, there would exist in D_1 a vertex not adjacent to any of u or v, contradicting $\alpha(D) = 2$.

First, assume that S contains a safe spanning tree T' and denote by C a strong terminal component of R. As $\lambda(D) \geq 2$, there exist at least two arcs xu and yv from C to S (with $x, y \in C$ and $u, v \in S$). If $u \neq v$, then there is an arc between u and v as S is semicomplete. Without loss of generality assume that uv is an arc of S and let e be the arc yv. If $u = v$ then we can choose arbitrarily $e = xu$ or $e = yv$. In both cases $T + T' + e$ is a safe spanning tree of D, contradicting the maximality of R.

So, S has no safe spanning tree and as S is semicomplete, it follows from Lemma 22, that $|V(S)| \leq 4$. We also have $|V(S)| > 1$, as a unique vertex always has a safe spanning tree. Thus, to conclude the proof of the Lemma, we have three cases to handle: $|V(S)| \in \{2, 3, 4\}$.

Assume first that S contains two vertices. Then, $D \setminus A(T)$ is strong, $D \setminus T$ has size two and is semicomplete. So, by Claim 22.1, D has a separating spanning tree, a contradiction again to the maximality of R.

Now assume that S contains three vertices, and denote by C a strong terminal component of R. As previously, as $\lambda(D) \geq 2$, there exist at least two arcs xu and yv from C to S (with $x, y \in C$ and $u, v \in S$). If $u \neq v$, as S is semicomplete there is an arc between u and v. Without loss of generality assume that uv is an arc of S and let e be the arc yv. If $u = v$ then we can choose arbitrarily $e = xu$ or $e = yv$. To conclude, denote $T + e$ by T' and notice that $D \setminus A(T')$ is strongly connected. As $D \setminus T'$ has size two and is semicomplete, by Claim 22.1, D has a separating spanning tree, a contradiction again to the maximality of R.

Finally, assume that S contains four vertices. As in the previous case, we can find an arc $e = zw$ with $z \in R$ and $w \in S$ such that $D \setminus (A(T) \cup \{e\})$ is strongly connected. As S as four vertices, w has in-degree or out-degree at least 2 in S. Assume that w has out-degree at least 2 and denote by u and v two out-neighbours of w in S such that uv is an arc of D. So, if we remove the arc wv from the digraph $D \setminus (A(T) \cup \{e\})$, the resulting digraph still contains the path wuv from w to u and so is still strongly connected. That is, if we denote by T' the tree $T + e + wv$, the digraph $D \setminus A(T')$ is strongly connected and $D - T'$ is semicomplete and has size two. Thus by Claim 22.1, D has a separating spanning tree, a contradiction again to the maximality of R. The case when w has in-degree at least 2 in S is analogous. □

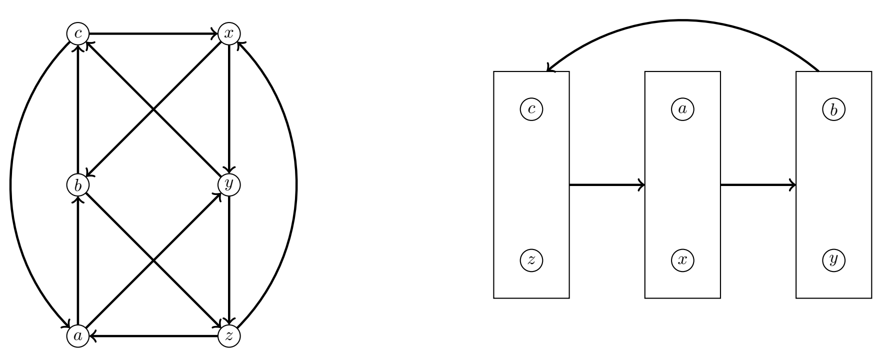

Figure 6: Two different drawings of the same 2-arc-strong co-bipartite digraph $\tilde{D}$ in which every spanning tree is separating.

Proposition 24. *The digraph $\tilde{D}$ in Figure 6 has no non-separating spanning tree.*

Proof. Note that H has 12 arcs and 6 vertices so if H would have a pair of arc-disjoint subdigraphs T, S where T is a spanning tree and S a strong spanning digraph, then $|A(S)| \leq 7$ must hold. This implies that S is either a hamiltonian cycle of H or it consist of a cycle C and a (C, C)-path P which picks up the remaining vertices of V. Now note that the only cycle lengths of D are 3 and 6. We now use that H has a number of automorphisms: there are 4 pairs of 3-cycles joined by a hamiltonian cycle on their vertices, namely $(abca, xyzx)$, $(ayza, bcxb)$, $(abza, cxyc)$, $(ayca, bzxb)$. Hence up to automorphisms there is only one hamiltonian cycle, namely $C_1 = abcxyza$. It is easy to check that $D \setminus A(C_1)$ is not connected. Hence, if T, S exist then we must have $|A(S)| = 7$ and S must consist of a 3-cycle C and a (C, C)-path P which picks up the remaining 3 vertices of V. By the symmetries above, we may assume that $C = abca$. Again, by permuting the vertices a, b, c if necessary, we can assume that P starts with the arc cx. This implies that $P = cxyza$ (as P picks up all the vertices x, y, z). Now we see that S contains the hamiltonian cycle $abcxyza$ and we saw above that removing the arcs of this cycle we disconnect the graph. □

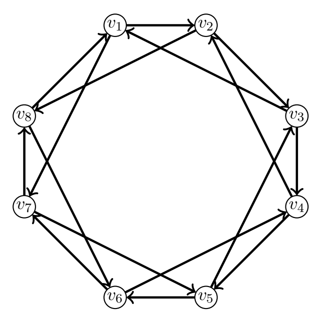

Figure 7: A 2-arc-strong digraph $\hat{D}$ with $\alpha(\hat{D}) = 2$ in which every spanning tree is separating.

Proposition 25. *The digraph $\hat{D}$ in Figure 7 has no non-separating spanning tree.*

Proof. As every vertex of $\hat{D}$ has in- and out-degree 2 we see that if T, S is a pair of arc-disjoint spanning subdigraphs of $\hat{D}$ such that T is connected and S is strongly connected, then T must be a hamiltonian path in $\hat{D}$ (as $d^+_S(x), d^-_S(x) \geq 1$ for every vertex $x \in V(\hat{D})$ and therefore $d^+_T(x), d^-_T(x) \leq 1$). Let $T = p_1 p_2 \dots p_8$. Let C denote the arcs on the hamiltonian cycle, $v_1 v_2 \dots v_8 v_1$ and let $\bar{C} = A(\hat{D}) \setminus C$. We first prove the following statement.

(i) $p_i p_{i+1}, p_{i+1} p_{i+2} \in C$ is not possible for any $i \in [6]$.

For the sake of contradiction, assume the above is true and without loss of generality that $p_i = v_1$. That is, $v_1 v_2 v_3$ is a subpath of T. Continuing along T out of v_3 and into v_1 we note that all arcs of T belong to C (i.e. we cannot use the arc $v_3 v_1$ in T, so the only possible arc out of v_3 is $v_3 v_4$ and the only possible arc into v_1 is $v_8 v_1$, etc.). However S would then contain two disjoint 4-cycles plus an extra arc, meaning it is not strongly connected, a contradiction.

As T is a hamiltonian path we note that $p_i p_{i+1} \in C$ for some $i \in \{2, 3, 4, 5\}$ (otherwise T contains a 4-cycle). By (i) we must have $p_{i-1} p_i, p_{i+1} p_{i+2} \in \bar{C}$. Without loss of generality assume that $p_i = v_4$, which implies that $v_6 v_4 v_5 v_3$ is a subpath of T. As T is a path (and $i \leq 5$) we must have that $v_6 v_4 v_5 v_3 v_1$ is a subpath of T (as $v_3 v_4 \notin A(T)$). This implies that the arcs $v_2v_3, v_3v_4, v_4v_2 \in A(S)$ (as $d^+_S(v_3), d^-_S(v_3), d^+_S(v_4), d^-_S(v_4) \geq 1$). As S has to contain arcs into and out of $\{v_2, v_3, v_4\}$ we must have $v_1v_2, v_2v_8 \in A(S)$. But now all arcs incident with v_2 are in S, a contradiction. □

By Proposition 25, the following Conjecture would be best possible in terms of the number of vertices.

Conjecture 26. Every digraph D on at least 9 vertices with $\lambda(D) = 2$ and $\alpha(D) \leq 2$ has a non-separating spanning tree

We provide below some support to Conjecture 26, by proving it for hamiltonian oriented graphs (ie. with no 2-cycle).

Theorem 27. *Every hamiltonian oriented graph $D = (V, A)$ on at least 9 vertices with $\lambda(D) \geq 2$ and $\alpha(D) = 2$ has a non-separating spanning tree*

Proof. Let C be a hamiltonian cycle of D and let $X_1, X_2, \ldots, X_k$ be the vertex sets of the connected components of $H = UG(D) \setminus A(C)$. If $k = 1$ we are done, so assume that $k \geq 2$. Note that each component has at least 3 vertices as $d_H(v) \geq 2$, and D contains no 2-cycle.

Suppose first that $k \geq 3$ and consider a vertex $v \in X_i$ for some $i \in [k]$. In $UG(D)$ the vertex v has at most 2 neighbours outside X_i. If v has no neighbours in some X_q, $q \neq i$, then let w be a non-neighbour of v in X_j, $j \notin \{i, q\}$ and let z be an arbitrary non-neighbour of w in X_q. Then $\{v, w, z\}$ is an independent set, contradiction. Therefore $k = 3$ and every vertex in X_i has a neighbour in each of the other sets X_j, implying that we can pick $x_i \in X_i$, $i \in [3]$ such that $\{x_1, x_2, x_3\}$ is an independent set, contradiction.

Hence $k = 2$ and we may assume that $|X_1| \geq |X_2|$. As $|V| \geq 9$ this implies that $|X_1| \geq 5$ and hence, as every arc between X_1 and X_2 belongs to C this implies that X_2 induces a complete subgraph of $UG(D)$ (every vertex of X_1 must be adjacent at least one of the vertices x, y in a pair of non-adjacent vertices $x, y \in X_2$ so if such a pair existed, at least one of x, y would be incident to 3 arcs of C, contradiction). Hence, by Theorem 23, we can assume that $|X_2| \leq 4$ and that $D[X_1]$ is not semicomplete. Note that for every pair of vertices $u, v \in X_1$ such that D has no arc between these, every vertex of X_2 has at least one edge to $\{u, v\}$ in $A(C)$. Also note that $\delta^0(D[X_i]) \geq 1$, $i \in [2]$ as only the arcs of C go between X_1 and X_2.

First suppose that $|X_2| = 4$. If X_1 contains vertices u_1, u_2, v_1, v_2 all distinct except possibly $u_2 = v_1$ so that there is no arc in D between u_i and v_i for $i = 1, 2$, then we get the contradiction that the undirected graph induced by $A(C)$ contains either a 4-cycle or an 8-cycle, contradicting the C is a hamiltonian cycle of D. Thus X_1 contains exactly one pair u, v of non-adjacent vertices and, by Theorem 23, we may assume that $|X_1| = 5$ so D has 9 vertices. As all 4 vertices of X_2 are adjacent to either u or v, exactly two of them are adjacent to u and the other two are adjacent to v. Now it is easy to see that C has at most one arc inside X_2 (otherwise C would contain a 3-cycle or a 6-cycle as a subdigraph). Consider first the case when C uses no arc inside X_2. Then we can label the vertices of V such that $X_1 = \{v_1, v_2, v_4, v_6, v_8\}$, $X_2 = \{v_3, v_5, v_7, v_9\}$ and $C = v_1v_2 \ldots v_9v_1$.

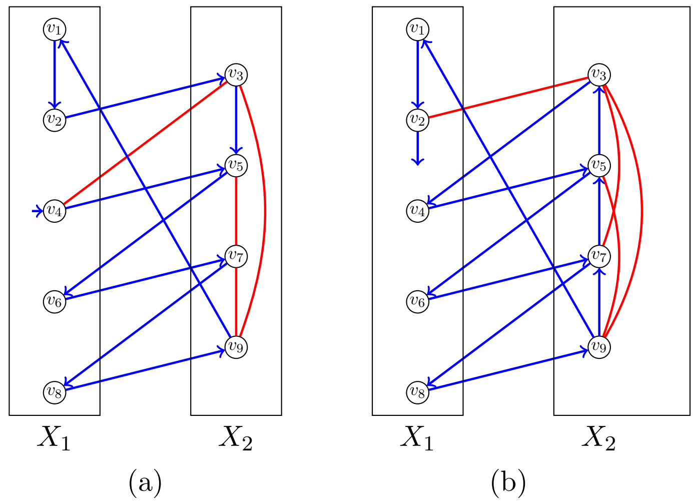

Figure 8: Illustrating two cases in the proof when $|X_2| = 4$. In (a) we illustrate the solution when v_3v_5 is an arc. The blue arcs form a strong spanning subdigraph S and the red edges, together with a spanning tree T' in $D[X_2]$, avoiding v_1v_2 and the blue arc into v_4 form a spanning tree T which is edge-disjoint from S. In (b) we indicate a solution when D contains the directed path $v_9v_7v_5v_3$.

Suppose first that D contains the arc v_3v_5. Then let wv_4 be an arbitrary arc entering v_4 in $D[X_1]$ (this exists as $\delta^0(D[X_1]) \geq 1$) and let T' be a spanning tree avoiding the arcs wv_4, v_1v_2 in $G' = UG(D[X_1])$. This tree exists as $G' \setminus \{wv_4, v_1v_2\}$ has 5 vertices and 7 edges and hence is connected. Then we obtain a strong spanning subdigraph S of D from C by deleting the arc v_3v_4 and adding the arcs v_3v_5, wv_4 and note that S is arc-disjoint from the spanning tree formed by T' and the edges of the path $v_4v_3v_9v_7v_5$ in $UG(D)$, see Figure 8(a).

Hence we can assume that $v_5v_3 \in A(D)$ and by an analogous argument we can assume that $v_7v_5, v_9v_7 \in A(D)$. Now let v_2w be an arbitrary arc leaving v_2 in $D[V(X_1)]$, let T'' be a spanning tree avoiding the arcs v_1v_2, v_2w in G' and let S' be the strong spanning subdigraph of D obtained from C by deleting the arc v_2v_3 and adding the arcs of the directed path $v_9v_7v_5v_3$ and the arc v_2w and note that S' is arc-disjoint from the spanning tree formed by T'' and the edges of the path $v_5v_9v_3v_7$ (in $UG(D)$) and the arc v_2v_3. See Figure 8(b).

Next we consider the case when C contains one arc of $D[V(X_2)]$. In this case, we may assume that v_5 and v_8 are the two vertices in X_1 that are non-adjacent in D and $X_2 = \{x_4, x_6, x_7, x_9\}$ such that $x_4x_5x_6$ and $x_7x_8x_9$ are subpaths of C. We may furthermore assume without loss of generality that $x_6x_7 \in A(C)$ (the case when $x_9x_4 \in A(C)$ is identical, by renaming vertices). This implies that we can label C as $C = v_1v_2 \dots v_9v_1$, and $X_1 = \{v_1, v_2, v_3, v_5, v_8\}$ and $X_2 = \{v_4, v_6, v_7, v_9\}$ (and v_5 and v_8 are non-adjacent in D).

If v_7v_9 is an arc of $D[V(X_2)]$, then let $j \in [3]$ be chosen such that v_j is an out-neighbour of v_8 in $D[V(X_1)]$ and let $j' \in [3] \setminus \{j\}$ be arbitrary. Now the strong spanning subdigraph S consisting of the cycle $v_1v_2 \dots v_7v_9v_1$ and the path $v_7v_8v_j$ is arc-disjoint from the spanning tree using the edges $v_4v_6, v_4v_7, v_4v_9, v_9v_8, v_8v_{j'}, v_5v_1, v_5v_2, v_5v_3$. Hence we can assume that v_9v_7 is an arc of $D[V(X_2)]$. A similar argument shows that we may assume that v_6v_4 is an arc of $D[V(X_2)]$. Now using that $\delta^0(D) \geq 2$ this implies that the remaining arcs in $D[V(X_2)]$ are v_7v_4, v_9v_6 and v_4v_9. See Figure 9(a).

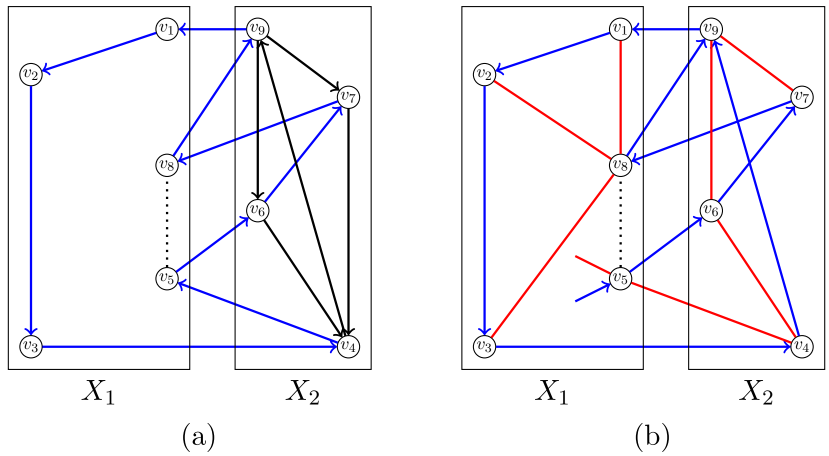

Figure 9: Illustrating the two last cases in the proof when $|X_2| = 4$.

Now choose $p \in [3]$ such that $v_p v_5$ is an arc of $D[V(X_1)]$ and let S' be the strong spanning subdigraph of D formed by the arcs of the cycle $v_1v_2v_3v_4v_9v_1$ and the path $v_pv_5v_6v_7v_8v_9$. Let $q \in [3] \setminus \{p\}$ be arbitrary and note that $D \setminus A(S')$ is connected as it contains the spanning tree on the edges $v_4v_5, v_4v_6, v_6v_9, v_7v_9, v_8v_1, v_8v_2, v_8v_3, v_5v_q$, see Figure 9(b). This completes the case when $|X_2| = 4$.

Consider now the case when $|X_2| = 3$. Recall that X_2 induces a 3-cycle in $UG(D)$. Let G be the complement of $UG[X_1]$. That is, $V(G) = V(X_1)$ and $uv \in E(G)$ if and only if u and v are non-adjacent in D. By Theorem 23 we may assume that $\alpha(G) \leq 4$ and as $\alpha(D) = 2$ we may assume that G contains no 3-cycle. As $|V(G)| \geq 6$ we note that we must therefore have a matching of size two in G. Let uv and $u'v'$ be two edges in a matching in G.

Note that at least three arcs between X_2 and $\{u, v\}$ belong to C (as otherwise there would be an independent set of size 3 containing u and v). There are also at least three arcs between X_2 and $\{u', v'\}$ in C. As $|X_2| = 3$ these 6 arcs are all the arcs between X_2 and X_1. Without loss of generality assume that u is incident to two arcs between X_1 and X_2 and u' is also incident to two arcs between X_1 and X_2, which implies that both v and v' are incident with exactly one arc between X_1 and X_2. As $\alpha(D) = 2$ and $\alpha(G) \leq 4$ we now note that $|X_2| = 6$ and $E(G) = \{uv, u'v'\}$ or $E(G) = \{uv, u'v', uu'\}$.

Let $u_8 = u$, $u_4 = v$, $u_6 = u'$ and $u_1 = v'$. We can now without loss of generality, label the vertices of V by $u_1, \ldots, u_9$ such that $X_1 = \{u_1, u_2, u_3, u_4, u_6, u_8\}$, $X_2 = \{u_5, u_7, u_9\}$, $C = u_1u_2 \ldots u_9u_1$ and there is no arc between u_1 and u_6 and no arc between u_4 and u_8. There may or may not be an arc between u_6 and u_8. See Figure 10.

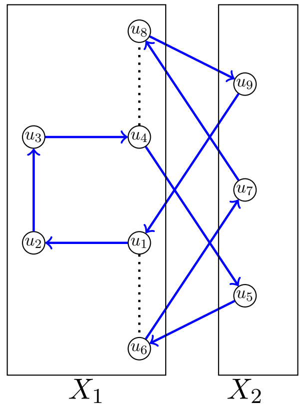

Figure 10: The hamiltonian cycle C in D when $|X_2| = 3$. The dotted edges indicate the two pairs of non-adjacent vertices in $D[X_1]$.

Moreover, as $\delta^+(D) \geq 2$ and D is oriented, we know that $D[X_2]$ is a directed 3-cycle. If D contains the arc u_7u_9, then as above we can find the desired pair S, T, see Figure 11(a).

Otherwise, it means that $D[X_2]$ is the directed 3-cycle $u_9u_7u_5u_9$, and then, as above we can find the desired pair S, T, see Figure 11(b).

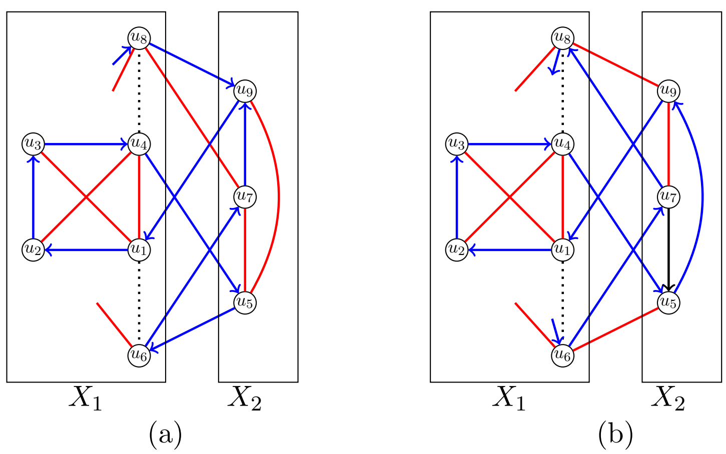

Figure 11: In (a): strong spanning subdigraph (in blue) and spanning tree (in red) when D contains the arc u_7u_5. In (b): strong spanning subdigraph (in blue) and spanning tree (in red) when D contains the 3-cycle $u_9u_7u_5u_9$

$\square$

6 Removing a hamiltonian path

Note that Theorem 15 implies that every 2-arc-strong semicomplete digraph D different from S_4 has an out-branching B^+ such that $D \setminus A(B^+)$ is strong. It is easy to check that S_4 also has such an out-branching. The purpose of this section is to prove that there exists 2-arc-strong digraphs with independence number 2 for which no hamiltonian path is non-separating.

For every natural number $r \geq 2$ let $T_r = (V, A)$ be the tournament with vertex set $\{u_0, u_1, \ldots, u_{r+1}, v_0, v_1, \ldots, v_{r+1}\}$ and arc set $\{u_{i-1}u_i | i \in [r]\} \cup \{v_i v_{i+1} | i \in [r]\} \cup \{u_i v_i | i \in [r]\} \cup \{v_1 v_0, v_0 u_0, v_0 u_1, u_0 v_1\} \cup \{u_{r+1} u_r, v_{r+1} u_{r+1}, u_r v_{r+1}, v_r u_{r+1}\}$ and for all remaining pairs not mentioned above the arcs goes from the vertex of higher index to the one with the lower index. See Figure 12.

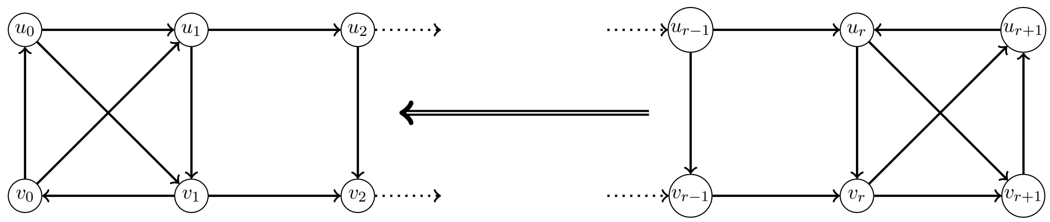

Figure 12: The tournament T_r. The fat arc in the middle indicates that all arcs not shown in the figure go from right to left.

Lemma 28. *For every $r \geq 4$ the tournament T_r is 2-arc-strong. Furthermore if P is a hamiltonian path in T_r starting in v_0 then v_0 cannot reach v_r in $T_r \setminus A(P)$.*

Proof. It is easy to check that v_0 has two arc-disjoint paths to every other vertex and that every vertex different from v_0 has two arc-disjoint paths to v_0. This implies that T_r is 2-arc-strong. For the sake of contradiction assume that there is a hamiltonian path, P, in T_r, and that v_0 can reach v_r in $S = T_r \setminus A(P)$. As for every $i \in [r-1]$ the two arcs $u_i u_{i+1}, v_i v_{i+1}$ form a 2-arc-cut of T_r seperating v_0 from v_r, one of these arcs must belong to S and the other to P. Similarly, as for every $i \in [r-2]$ the two arcs $u_i u_{i+1}, v_{i+1} v_{i+2}$ form a 2-arc-cut of T_r seperating v_0 from v_r one of these arcs must belong to S and the other to P. Let $A_1 = \{u_1 u_2, u_2 u_3, \ldots, u_{r-1} u_r\}$ and let $A_2 = \{v_1 v_2, v_2 v_3, \ldots, v_{r-1} v_r\}$, and note that by the previous argument we must have $A_i \subseteq A(S)$ and $A_{3-i} \subseteq A(P)$ for some $i \in [2]$. Without loss of generality assume that $A_1 \subseteq A(S)$ and $A_2 \subseteq A(P)$, which implies that P cannot contain both u_2 and u_3, a contradiction to the existence of P and S. □

The following corollary follows immediatly from Lemma 28.

Corollary 29. *For every $r \geq 4$ the tournament T_r is 2-arc-strong and for every hamiltonian path P starting in the vertex v_0 the digraph $D \setminus A(P)$ is not strongly connected.*

Theorem 30. *There exist infinitely many 2-arc-strong digraphs D with $\alpha(D) = 2$ such that deleting the arcs of any hamiltonian path leaves a non-strong digraph.*

Proof. For each $r \geq 4$ let T_r be the 2-arc-strong tournament defined in Lemma 28 and form the digraph D_r from two copies T_r^1, T_r^2 of T_r (whose vertices are superscripted) by adding two arbitrary arcs from $V(T_r^i)$ to v_0^{3-i} for $i = 1, 2$. Since each T_r is a tournament, we have $\alpha(D_r) = 2$. Moreover, as T_r^1 and T_r^2 arc 2-arc strong, D_r is 2-arc strongly connected also. Suppose that D_r has a hamiltonian path P such that $D \setminus A(P)$ is strong. Without loss of generality P starts in $V(T_r^1)$ and thus the restriction of P to $V(T_r^2)$ is a hamiltonian path P' starting at v_0^2. By Lemma 28 we note that v_0^2 cannot reach v_r^2 in $T_r^2 \setminus A(P')$, which implies that no vertex in T_r^1 can reach v_r^2 in $D_r \setminus A(P)$. So $D_r \setminus A(P)$ is not strong, a contradiction. □

7 Non-separating hamiltonian paths in graphs with independence number 2

In contrast to the result in Theorem 30 above, for the case of undirected graphs of independence number 2 we have the following positive result on non-separating hamiltonian paths.

Theorem 31. *Let G be a 2-edge-connected graph with $\delta(G) \geq 4$ and $\alpha(G) \leq 2$. Then, G contains a spanning tree and a Hamiltonian path which are edge-disjoint.*

Proof. Let G be a 2-edge connected graph with $\delta(G) \geq 4$ and $\alpha(G) \leq 2$. It is easy to see that every connected graph with independence number at least 2 has a spanning tree with a number of leaves at most its independence number. Hence G contains a Hamiltonian path P. If AY$G \setminus E(P)$ is connected, we are done. Otherwise let $X_1, X_2, \dots, X_p$ be the connected components of $G \setminus E(P)$. Notice that as $\delta(G) \geq 4$ and P is a path, we have $\delta(G \setminus E(P)) \geq 2$, and in particular, we have $|X_i| \geq 3$ for all $i = 1, \dots, p$. If $p \geq 3$, consider u an extremity of P and assume without loss of generality that $u \in X_1$ and that its neighbour v in P is in $X_1 \cup X_2$. It means that all the vertices of $X_2 \setminus \{v\}$ and X_3 are non neighbours of u and hence must form a complete subgraph of G. In particular, all the edges between $X_2 \setminus \{v\}$ and X_3 are edges of P, which is not possible as $|X_2 \setminus \{v\}| \geq 2$ and $|X_3| \geq 3$, implying that P would contain a cycle. So, $G \setminus E(P)$ contains exactly the two connected components X_1 and X_2.

Notice that the case $|X_1| = |X_2| = 3$ is not possible, as in this case, as $\delta(G) \geq 4$, every vertex of X_1 would have at least two neighbours in X_2 and every vertex of X_2 would have at least two neighbours in X_1, and so P would contain a cycle, which is not possible. Hence $\max\{|X_1|, |X_2|\} \geq 4$ and we may assume that $|X_1| \geq 4$.

Assume first that $\alpha(G[X_1]) = \alpha(G[X_2]) = 1$, that is, they are both complete graphs. In this case, we will show how to build a Hamiltonian path and a spanning tree of G which are edge-disjoint. If x is a vertex of any complete graph K on at least 4 vertices, it is easy to find a Hamiltonian path which starts in x and a spanning tree which are edge-disjoint. Suppose first that $|X_2| \geq 4$. Then it follows from the fact that G is 2-edge connected, that there exist two distinct edges x_1x_2 and y_1y_2 with $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Now consider for $i = 1, 2$ a Hamiltonian path P_i of $G[X_i]$ starting in x_i and a spanning tree T_i of $G[X_i]$ edge-disjoint from P_i. We conclude by considering the Hamiltonian path $(P_1 \cup P_2) + x_1x_2$ of G edge-disjoint from the spanning tree $(T_1 \cup T_2) + y_1y_2$ of G. Hence we may assume that $|X_2| = 3$ and denote the vertices in X_2 by $\{x_2, y_2, z_2\}$. As $\delta(G) \geq 4$, there exist three distinct edges of G x_2x_1, y_2y_1, z_2z_1 such that x_1, y_1 and z_1 belong to X_1. So, we consider a Hamiltonian path P_1 of $G[X_1]$ starting in x_1 and a spanning tree T_1 of $G[X_1]$ edge-disjoint from P_1. We conclude then with the Hamiltonian path $P' = (P_1 \cup x_2y_2z_2) + x_2x_1$ of G and the spanning tree $T_1 + x_2z_2 + y_2y_1 + z_2z_1$ of G edge-disjoint from P.

If $\alpha(G[X_1]) = 1$ and $\alpha(G[X_2]) = 2$, then as $\delta(G) \geq 4$, we must have $|X_2| \geq 4$. In this case we may swap the names of X_1 and X_2, which implies that we may assume without loss of generality that $\alpha(G[X_1]) = 2$. Let x_1 and y_1 be two vertices of X_1 which are not adjacent in G. As every vertex of X_2 is adjacent to x_1 or y_1 in G and as the corresponding edges must be edges of P, we have $|X_2| \leq 4$. Suppose $|X_2| = 4$. Then we will prove that $G[X_1]$ is almost complete, that is it contains all the possible edges except x_1y_1. Denote the vertices of X_2 by $\{x_2, y_2, z_2, t_2\}$, and as $\alpha(G) = 2$, we can assume that x_2 and y_2 are adjacent to x_1 and that z_2 and t_2 are adjacent to y_1. Assume first that $G[X_1]$ contains two non-adjacent vertices z_1 and t_1 both distinct from x_1 and y_1. As $\alpha(G) = 2$, the vertex x_2 has to be adjacent to z_1 or t_1. We can assume that z_1x_2 is an edge of G, but then y_2 has to be adjacent to t_1 as there is no cycle induced by the edges of P. Similarly, we can assume that z_1z_2 is an edge of P, but then t_2 cannot be adjacent to any of z_1, t_1 without creating a cycle induced by the edges of P. So, $\{z_1, t_1, t_2\}$ is an independent set, contradicting $\alpha(G) = 2$. Now assume that X_1 contains a vertex $z_1 \neq y_1$ which is non-adjacent to x_1. Then z_2 has to be adjacent to x_1 or z_1 and as x_2 and y_2 are already adjacent to x_1 in P, z_2 must be adjacent to z_1. Similarly, t_2 is adjacent to z_1, but then $z_1z_2y_1t_2$ would form a cycle with the edges of P, a contradiction. Similarly we can prove that every vertex of X_1 except x_1 is adjacent to y_1 so $G[X_1]$ is the graph $K_{|X_1|} - x_1y_1$. As $|X_1| \geq 4$, it is easy to find then in $G[X_1]$ two vertex-disjoint paths P_1 and P_1' such that $V(P_1) \cup V(P_1') = X_1$, the path P_1 ends in x_1 and the path P_1' ends in y_1 and $G[X_1] - P_1 - P_2$ is connected and so contains a spanning tree T_1. On the other hand, as $\delta(G[X_2] - P) \geq 2$, the graph $G[X_2]$ contains a 4-cycle. One of the edges of this 4-cycle goes from $\{x_2, y_2\}$ to $\{z_2, t_2\}$. So, denote this 4-cycle by $abcda$ such that a is adjacent to x_1 and d is adjacent to y_1. Now, we consider the Hamiltonian path $P' = (P_1 \cup P_1') + ab + bc + cd$ of G. As b and c have at least one neighbour each in X_1, we know that $G \setminus E(P')$ has at most 2 connected components, one containing $X_1 \cup \{b, c\}$, the other one containing $\{a, d\}$. But as $\delta(G) \geq 4$, a has a neighbour in $X_1 \cup \{b, c\}$ different from x_1, and so, $G \setminus E(P')$ is connected.

The only remaining case is when $|X_2| = 3$. Denote the vertices of X_2 by $\{x_2, y_2, z_2\}$ and assume without loss of generality that x_2 and y_2 are adjacent to x_1 and that z_2 is adjacent to y_1. As z_2 as degree at least 4 in G, there exist a vertex $z_1 \in X_1$ distinct from x_1 and y_1 such that z_1 is a neighbour of z_2 in P. More generally, as $\delta(G) \geq 4$, every vertex of X_2 has exactly two neighbours in X_1, which are then neighbours in P, $G[X_2]$ is a complete graph and finally every vertex of X_2 has degree exactly 4. In particular, $\{x_1, z_2\}$ is an independent set of size 2, and every vertex of $X_1 \setminus \{x_1, y_1, z_1\}$ is adjacent to x_1. Now, let us focus on the extremities of the path P. Both cannot lie in $Y = \{x_1, y_1, z_1, x_2, y_2, z_2\}$ as the only vertices of Y with degree less than 2 in P are y_1 and z_1, and they cannot be both extremities of P as otherwise, P would be the path $y_1z_2z_1$. So, denote by p an extremity of P not lying in Y and recall that x_1 is adjacent to every vertex of $V \setminus Y$ so x_1p is an edge of G. Now consider the Hamiltonian path P' of G defined by $P' = (P - x_2x_1 - x_1y_2) + x_2y_2 + x_1p$. To conclude, let us prove that $G \setminus E(P')$ is connected. Indeed, every vertex of $V \setminus \{y_1, z_1, z_2\}$ is linked to x_1, and all the corresponding edges except px_1 are edges of $G \setminus E(P')$. So, $G \setminus E(P')$ induces a connected graph on $Z = V \setminus \{y_1, z_1, z_2, p\}$. Moreover, z_2 is adjacent to x_2 and x_2z_2 is not an edge of P'. And by choice, p is not adjacent to z_2 and so has a neighbour in Z different from x_1. Finally, y_1 and z_1 have both at least one neighbour in $G \setminus E(P')$ which belongs to $V \setminus \{y_1, z_1\}$. Thus, $G \setminus E(P')$ is connected and the proof is complete. □

Notice that we cannot replace $\delta(G) \geq 4$ by $\delta(G) = 3$ (even if $\lambda(G) = 3$) as shown by the graph built from two 3-cycles linked by a perfect matching. Also, any 3-regular graph, G, has $|E(G)| = 3|V(G)|/2$, so cannot contain two edge-disjoint spanning trees when $|V(G)| > 4$, and therefore also not a hamiltonian path and a spanning tree that are edge-disjoint.

8 Remarks and open problems

All proofs in this paper are constructive and it is not difficult to derive polynomial algorithms for finding the desired objects in case they exist. We leave the details to the interested reader.

Problem 32. Determine the complexity of deciding whether a strong digraph of independence number 2 has a non separating out-branching.

Problem 33. Determine the complexity of deciding whether a strong digraph of independence number 2 has a non separating spanning tree.

This problem is NP-complete for general digraphs as shown in [8].

Theorem 23 suggests that perhaps we can get rid of the requirement on the minimum in-degree in Theorem 8 when the digraph has enough vertices.

Conjecture 34. There exists an integer K such that every digraph D on at least K vertices with $\lambda(D) \geq 2$ and $\alpha(D) = 2$ has a non-separating out-branching.

It is not difficult to check that every member of the infinite class of digraphs that we used in Proposition 12 has a non-separating branching from every vertex.

Conjecture 35. There exists an integer L such that every digraph D on at least L vertices with $\lambda(D) \geq 2$ and $\alpha(D) = 2$ has a non-separating out-branching B_s^+ for every choice of $s \in V$.

Question 36. Does every 3-arc-strong digraph D with $\alpha(D) = 2$ have a pair of arc-disjoint spanning strong subdigraphs?

In Proposition 12 we showed that 2-arc-strong connectivity and high minimum semi-degree is not enough to guarantee such digraphs.

References

[1] J. Bang-Jensen. Edge-disjoint in- and out-branchings in tournaments and related path problems. *J. Combin. Theory Ser. B*, 51(1):1–23, 1991.

[2] J. Bang-Jensen, S. Bessy, F. Havet, and A. Yeo. Arc-disjoint in- and out-branchings in digraphs of independence number at most 2. *submitted*, 2020.

[3] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer-Verlag, London, 2nd edition, 2009.

[4] J Bang-Jensen, G. Gutin, and A. Yeo. Arc-disjoint strong spanning subdigraphs of semicomplete compositions. *J. Graph Theory*, to appear.

[5] J. Bang-Jensen, F. Havet, and A. Yeo. Spanning eulerian subdigraphs in tournaments. *submitted*, 2019.

[6] J. Bang-Jensen and S. Simonsen. Arc-disjoint paths and trees in 2-regular digraphs. *Disc. Appl. Mathematics*, 161:2724–2730, 2013.

[7] J. Bang-Jensen and A. Yeo. Decomposing k-arc-strong tournaments into strong spanning subdigraphs. *Combinatorica*, 24(3):331–349, 2004.

[8] J. Bang-Jensen and A. Yeo. Arc-disjoint spanning sub(di)graphs in digraphs. *Theor. Comput. Sci.*, 438:48–54, 2012.

[9] P. Camion. Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris*, 249:2151–2152, 1959.

[10] C.C. Chen and P. Manalastas, Jr. Every finite strongly connected digraph of stability 2 has a Hamiltonian path. *Discrete Math.*, 44(3):243–250, 1983.

[11] M. Chudnovsky, A. Scott, and P.D. Seymour. Disjoint paths in unions of tournaments. *J. Comb. Theory, Ser. B*, 135:238–255, 2019.

[12] J. Edmonds. Edge-disjoint branchings. In *Combinatorial Algorithms*, pages 91–96. Academic Press, 1973.

[13] A. Fradkin and P.D. Seymour. Edge-disjoint paths in digraphs with bounded independence number. *J. Combin. Theory Ser. B*, 110:19–46, 2015.

[14] L. Lovász. On two min–max theorems in graph theory. *J. Combin. Theory Ser. B*, 21:96–103, 1976.

[15] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combinatorics*, 1, 1994. Revision 15: March 3, 2017.

[16] C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. *Annals of the New York Academy of Sciences*, 555:402–412, 1989.